STAT 576 HOMEWORK 4

DUE DEC. 13, 2024 (FRIDAY), 11:59 PM PST

NOTES. **NO** late submission will be accepted except for approval from the instructor. Answers should be either scanned or typed. For the coding part (if any), you may either prepare an R notebook or put the code in a separate file.

2D Ising Model

In this homework, we will investigate the phase transition phonomenon in the 2D Ising model through Markov Chain Monte Carlo. The 2D Ising system consists of a lattice of spins $s_{ij} \in \{-1, 1\}$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The energy of the system is given by

$$E(\boldsymbol{s}) = -J \sum_{i,j} \sum_{(i',j') \in \mathcal{N}_{ij}} s_{ij} s_{i'j'},$$

where $\mathcal{N}_{ij}$ is the set of nearest neighbors of (i, j), and J is the coupling constant. For example, if $(i, j) = (3, 3)$, then $\mathcal{N}_{33} = \{(2, 3), (4, 3), (3, 2), (3, 4)\}$. If $(i, j) = (1, 2)$, then $\mathcal{N}_{12} = \{(1, 1), (1, 3), (2, 2)\}$. This energy contains all the pairwise insteractions between the spins.

The distribution of the spins is given by the Boltzmann distribution

$$p_T(\boldsymbol{s}) \propto \exp\left\{-\frac{E(\boldsymbol{s})}{kT}\right\},$$

where T is the temperature and k is the Boltzmann constant. The average magnetization is given by

$$\mathcal{M}(T) = \left| \mathbb{E}_{p_T} \left[\frac{1}{mn} \sum_{i,j} s_{ij} \right] \right|$$

is the average spins of the system under the Boltzmann distribution at temperature T.

(a) For simplicity, we set $k = 1$ and $J = 5$. What is the conditional distribution of s_{ij} given all the other spins?

(b) For $m = n = 16$ and $T = 5$, write a MCMC program to sample from the Boltzmann distribution and estimate the average magnetization $\mathcal{M}_5$. (Hint: you can use the Gibbs sampler to sample from the distribution.)

(c) It is a well-known fact that the 2D Ising model exhibits a phase transition at some critical temperature T_c. The average magnetization $\mathcal{M}(T)$ is expected to be zero for $T > T_c$ and non-zero for $T < T_c$. Repeat part (b) for different temperatures between $T = 1$ and $T = 25$ and plot the average magnetization as a function of temperature. What do you observe?