STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 9: Multifactor Analysis of Variance II

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Two-way ANOVA with Replication

Now we consider the two-way ANOVA with replication. That is for each combination of the two factors, we have more than one observation.

$$X_{ijk} = \mu_{ij} + \epsilon_{ijk},$$

where the index ijk denotes the kth observation in the ith level of factor A and the jth level of factor B.

For simplicity, we assume $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, J$, and $k = 1, 2, \ldots, K$ — equal number of observations in each cell.

Two-way ANOVA with Replication

We rewrite μ_{ij} as in the additive model but with an additional term for the interaction effect:

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}.$$

- The term γ_{ij} is called the **interaction effect**.
- We don't include the interaction term in two-way ANOVA without replication because we can't estimate it (more parameters than observations).
- With replication, we can estimate the interaction effect.
- The number of parameters for μ_{ij} is IJ. So we need to show that the number of above model is IJ as well.

Two-way ANOVA with Replication

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}.$$

- The number of parameters is $1 + I + J + IJ = (I+1)(J+1)$.
- The constraints are:

$$\sum_i \alpha_i = 0, \qquad \sum_j \beta_j = 0,$$

$$\sum_i \gamma_{ij} = 0 \quad \text{for all } j = 1, \ldots, J, \qquad \sum_j \gamma_{ij} = 0 \quad \text{for all } i = 1, \ldots, I.$$

  The number of constraints is $I + J + 2$.
  The number of **independent** constraints is $I + J + 1$.
- Therefore, the effective number of parameters is $(I+1)(J+1) - (I+J+1) = IJ$.

Two-way ANOVA with Replication

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}.$$

- α_i: the (main) effect of the ith level of factor A.
- β_j: the (main) effect of the jth level of factor B.
- γ_{ij}: the interaction effect between the ith level of factor A and the jth level of factor B.

Sample Means

Similar to the one-way ANOVA, we can calculate the sample means for each level of the two factors and the interaction effect:

$$\bar{X}_{i\cdot\cdot} = \frac{1}{JK}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk}, \qquad \bar{X}_{\cdot j\cdot} = \frac{1}{IK}\sum_{i=1}^{I}\sum_{k=1}^{K} X_{ijk}$$

$$\bar{X}_{ij\cdot} = \frac{1}{K}\sum_{k=1}^{K} X_{ijk}, \qquad \bar{X}_{\cdot\cdot\cdot} = \frac{1}{IJK}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk}.$$

- $\bar{X}_{i\cdot\cdot}$: the sample mean for the ith level of factor A.
- $\bar{X}_{\cdot j\cdot}$: the sample mean for the jth level of factor B.
- $\bar{X}_{ij\cdot}$: the sample mean for the ijth cell.
- $\bar{X}_{\cdot\cdot\cdot}$: the grand mean.

Estimation

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}.$$

Using method of moments, the estimators should satisfy

$$\bar{X}_{ij\cdot} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij} \quad (*)$$

- Averaging $(*)$ over all i and j gives: $\bar{X}_{\cdots} = \hat{\mu}$.
- Averaging $(*)$ over all i and fixing j gives: $\bar{X}_{\cdot j\cdot} = \hat{\mu} + \hat{\beta}_j \Longrightarrow \hat{\beta}_j = \bar{X}_{\cdot j\cdot} - \bar{X}_{\cdots}$.
- Similarly, $\hat{\alpha}_i = \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdots}$.
- By plugging in the above estimators, we can solve for $\hat{\gamma}_{ij} = \bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdots}$.

Estimation

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}.$$

The estimators are:

$$\begin{aligned}
\hat{\mu} &= \bar{X}_{...}, \\
\hat{\alpha}_i &= \bar{X}_{i..} - \bar{X}_{...}, \\
\hat{\beta}_j &= \bar{X}_{.j.} - \bar{X}_{...}, \\
\hat{\gamma}_{ij} &= \bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{...} \\
\hat{\epsilon}_{ijk} &= X_{ijk} - \bar{X}_{ij.}.
\end{aligned}$$

Sum of Squares

$$X_{ijk} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij} + \hat{\epsilon}_{ijk}.$$

Now we can define the sum of squares for each term in the model:

$$SSE = \sum_i \sum_j \sum_k \hat{\epsilon}_{ijk}^2 = \sum_i \sum_j \sum_k (X_{ijk} - \bar{X}_{ij\cdot})^2, \qquad df = IJ(K-1)$$

$$SSA = \sum_i \sum_j \sum_k \hat{\alpha}_i^2 = JK \sum_i (\bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot\cdot\cdot})^2, \qquad df = I-1$$

$$SSB = \sum_i \sum_j \sum_k \hat{\beta}_j^2 = IK \sum_j (\bar{X}_{\cdot j\cdot} - \bar{X}_{\cdot\cdot\cdot})^2, \qquad df = J-1$$

$$SSAB = \sum_i \sum_j \sum_k \hat{\gamma}_{ij}^2 = K \sum_i \sum_j (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\cdot\cdot\cdot})^2, \qquad df = (I-1)(J-1)$$

$$SST = \sum_i \sum_j \sum_k (X_{ijk} - \bar{X}_{\cdot\cdot\cdot})^2. \qquad df = IJK-1$$

Also, we have $SST = SSA + SSB + SSAB + SSE$.

Mean Squares

We can define the mean squares and give their expected values:

$$MSE = \frac{SSE}{IJ(K-1)} \qquad E(MSE) = \sigma^2,$$

$$MSA = \frac{SSA}{I-1} \qquad E(MSA) = \sigma^2 + \frac{JK}{I-1}\sum_i \alpha_i^2,$$

$$MSB = \frac{SSB}{J-1} \qquad E(MSB) = \sigma^2 + \frac{IK}{J-1}\sum_j \beta_j^2,$$

$$MSAB = \frac{SSAB}{(I-1)(J-1)} \qquad E(MSAB) = \sigma^2 + \frac{K}{(I-1)(J-1)}\sum_i \sum_j \gamma_{ij}^2.$$

Testing

To test the main effect of factor A, we consider the following hypotheses:

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0,$$
$$H_1 : \text{at least one } \alpha_i \neq 0.$$

Under null the model is

$$X_{ijk} = \mu + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

Under alternative the model is

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

We summarize the two nested models as

Model	SS Model	SS Error	Difference in SS Error	Difference in df
Full	SSA+SSB+SSAB	SSE		
Reduced	SSB + SSAB	SSE+SSA	SSA	I-1

Testing

Model	SS Model	SS Error	Difference in SS Error	Difference in df
Full	SSA+SSB+SSAB	SSE		
Reduced	SSB + SSAB	SSE+SSA	SSA	I-1

Therefore, the F-test should be

$$F_A = \frac{SSA/(I-1)}{SSE/[IJ(K-1)]} = \frac{MSA}{MSE} \sim F_{I-1,IJ(K-1)} \text{(under null)}$$

We should reject null when $F_A > F_{\alpha,I-1,IJ(K-1)}$.

Testing

Similarly, to test the main effect of factor B, we consider the following hypotheses:

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_J = 0,$$
$$H_1 : \text{at least one } \beta_j \neq 0.$$

The F-test should be

$$F_B = \frac{MSB}{MSE} \sim F_{J-1,IJ(K-1)} \text{(under null)}$$

We should reject null when $F_B > F_{\alpha,J-1,IJ(K-1)}$.

Testing

To test the interaction effect, we consider the following hypotheses:

$$
\begin{aligned}
H_0 :& \gamma_{11} = \gamma_{12} = \cdots = \gamma_{IJ} = 0, \\
H_1 :& \text{at least one } \gamma_{ij} \neq 0.
\end{aligned}
$$

The F-test should be

$$F_{AB} = \frac{MSAB}{MSE} \sim F_{(I-1)(J-1),IJ(K-1)} \text{(under null)}$$

We should reject null when $F_{AB} > F_{\alpha,(I-1)(J-1),IJ(K-1)}$.

Questions: What are the full model and reduced model in this case?

ANOVA Table

The ANOVA table for the two-way ANOVA with replication is:

Source	SS	df	MS	F
Factor A	SSA	$I-1$	MSA	F_A
Factor B	SSB	$J-1$	MSB	F_B
Interaction	SSAB	$(I-1)(J-1)$	MSAB	F_{AB}
Error	SSE	$IJ(K-1)$	MSE	
Total	SST	$IJK-1$		

Example

Data: thermal properties of asphalt mix under three different binder grades and three different coarse aggregate contents.

	Coarse Aggregate Content (%)			
Asphalt Binder Grade	**38**	**41**	**44**	$\bar{x}_{i\cdot\cdot}$
PG58	.835, .845	.822, .826	.785, .795	.8180
PG64	.855, .865	.832, .836	.790, .800	.8297
PG70	.815, .825	.800, .820	.770, .790	.8033
$\bar{x}_{\cdot j\cdot}$	.8400	.8227	.7883	

Example

The ANOVA table is:

Source	DF	SS	MS	f	P
AsphGr	2	.0020893	.0010447	14.12	0.002
AggCont	2	.0082973	.0041487	56.06	0.000
Interaction	4	.0003253	.0000813	1.10	0.414
Error	9	.0006660	.0000740		
Total	17	.0113780			

Mixed Effect Models

In some experiments, one or more factors are randomly selected from a larger population.

- The factors that are pre-selected are called **fixed effects**.
- The factors that are randomly selected are called **random effects**.
- If a multi-factor model contains both fixed and random effects, it is called a **mixed effect model**.

Mixed Effect Models

Two-way ANOVA model with two fixed effects (with replication):

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}.$$

- α_i and β_j's are unknown fixed parameters.

Now we let the second factor to be random:

$$X_{ijk} = \mu + \alpha_i + B_j + G_{ij} + \epsilon_{ijk}.$$

- B_j is the random effect of the jth level of factor B.
- G_{ij} is the interaction effect between the ith level of factor A and the jth level of factor B.

Mixed Effect Models

$$X_{ijk} = \mu + \alpha_i + B_j + G_{ij} + \epsilon_{ijk}.$$

Certain assumptions are made for the random effects:

- $B_j \sim N(0, \sigma_B^2)$.
- $G_{ij} \sim N(0, \sigma_G^2)$.

Mixed Effect Models

$$X_{ijk} = \mu + \alpha_i + B_j + G_{ij} + \epsilon_{ijk}.$$

Now we consider the constraints:

- $\sum_i \alpha_i = 0$ — because it's on the fixed effect, so we keep it.
- $\sum_j B_j = 0$ — we should **not** assume it because of random sampling.
- $\sum_j G_{ij} = 0$ — we should **not** assume it as well.
- $\sum_i G_{ij} = 0$ — it depends.

Mixed Effect Models

$$X_{ijk} = \mu + \alpha_i + B_j + G_{ij} + \epsilon_{ijk}.$$

The **unrestricted model** assumes:

- $\epsilon_{ijk} \sim N(0, \sigma^2)$.
- $B_j \sim N(0, \sigma_B^2)$.
- $G_{ij} \sim N(0, \sigma_G^2)$.
- $\sum_i \alpha_i = 0$.
- B_j and G_{ij} are independent of each other.

Mixed Effect Models

$$X_{ijk} = \mu + \alpha_i + B_j + G_{ij} + \epsilon_{ijk}.$$

The **restricted model** assumes:

- $\epsilon_{ijk} \sim N(0, \sigma^2)$.
- $B_j \sim N(0, \sigma_B^2)$.
- $G_{ij} \sim N\left(0, \frac{I-1}{I}\sigma_G^2\right)$.
- $\sum_i \alpha_i = 0$.
- $\sum_i G_{ij} = 0$.
- Now G_{ij}'s are correlated.

Mixed Effect Models

The direct consequences of the different assumptions are the expected MS values.

Unrestricted Model

$$\begin{aligned} E(MSE) &= \sigma^2 \\ E(MSA) &= \sigma^2 + K\sigma_G^2 + \frac{JK}{I-1}\sum_i \alpha_i^2 \\ E(MSB) &= \sigma^2 + K\sigma_G^2 + IK\sigma_B^2 \\ E(MSAB) &= \sigma^2 + K\sigma_G^2 \end{aligned}$$

Restricted Model

$$\begin{aligned} E(MSE) &= \sigma^2 \\ E(MSA) &= \sigma^2 + K\sigma_G^2 + \frac{JK}{I-1}\sum_i \alpha_i^2 \\ E(MSB) &= \sigma^2 + IK\sigma_B^2 \\ E(MSAB) &= \sigma^2 + K\sigma_G^2 \end{aligned}$$

Mixed Effect Models

The further consequence is that we should use different F-values for the tests.

Test (H_0)	Unrestricted Model	Restricted Model
$\alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$	$F = MSA/MSAB$	$F = MSA/MSAB$
$\sigma_B^2 = 0$	$F = MSB/MSAB$	$F = MSB/MSE$
$\sigma_G^2 = 0$	$F = MSAB/MSE$	$F = MSAB/MSE$