STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 8: Multifactor Analysis of Variance I

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Multifactor ANOVA

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- In this lecture, we will discuss multifactor ANOVA, which is used to compare the means of two or more groups when there are two or more factors.

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Topics to be covered:

- Two-factor ANOVA without replication
- Two-factor ANOVA with replication

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Topics to be covered:

- Two-factor ANOVA without replication
- Two-factor ANOVA with replication

We will focus on:

- Sum of squares and mean squares
- F-test and multiple comparisons
- Fixed and random effects

Two-Factor ANOVA without Replication

Suppose we have two factors:

- Factor A: I levels, $i = 1, 2, \ldots, I$
- Factor B: J levels, $j = 1, 2, \ldots, J$

The possible number of treatments is $I \times J$.
Remark: different index notations from one-way ANOVA.

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Furthermore, we assume we have **one** observation for each treatment:

- X_{ij} is the observation for the i-th level of factor A and the j-th level of factor B.
- x_{ij} the observed value of X_{ij}.

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The analysis of variance for such a model is called a **two-factor ANOVA without replication**.

Example

A research on the erabsiablity of stains on a fabric from three brands of pen and four different washing treatments. The observation is quantitaive measurement of color change.

		Washing Treatment					
		1	**2**	**3**	**4**	**Total**	**Average**
Brand of Pen	**1**	.97	.48	.48	.46	2.39	.598
	2	.77	.14	.22	.25	1.38	.345
	3	.67	.39	.57	.19	1.82	.455
	Total	2.41	1.01	1.27	.90	5.59	
	Average	.803	.337	.423	.300		.466

The Means

Similar to one-way ANOVA, we can define the following means:

- The sample mean of the i-th level of factor A:

$$\bar{X}_{i\cdot} = \frac{1}{J}\sum_{j=1}^{J} X_{ij}$$

- The sample mean of the j-th level of factor B:

$$\bar{X}_{\cdot j} = \frac{1}{I}\sum_{i=1}^{I} X_{ij}$$

- The grand sample mean:

$$\bar{X}_{\cdot\cdot} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij}$$

Fixed Effect Model

In a fixed effect model, we assume

$$X_{ij} = \mu_{ij} + \varepsilon_{ij} \quad \text{for } i = 1, 2, \ldots, I, j = 1, 2, \ldots, J$$

with $\varepsilon_{ij} \sim N(0, \sigma^2)$.

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- Total number of observations: IJ
- Total number of parameters: $IJ + 1$
- The model is not estimable until we impose extra constraints.

Fixed Effect Model

We consider the following **additive model**:

$$X_{ij} = \alpha_i + \beta_j + \varepsilon_{ij},$$

that is, we assume $\mu_{ij} = \alpha_i + \beta_j$.

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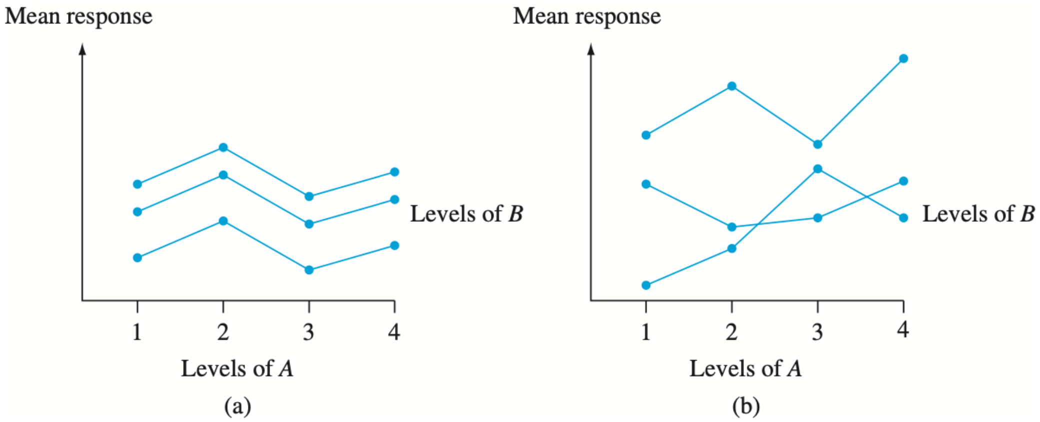
The difference in responeses between any two treatment levels can be decomposed into the sum of the differences of the corresponding factor levels:

$$\mu_{ij} - \mu_{i'j'} = (\alpha_i - \alpha_{i'}) + (\beta_j - \beta_{j'})$$

Fixed Effect Model

The additivity assumption can be checked visually.



Fixed Effect Model

$$X_{ij} = \alpha_i + \beta_j + \varepsilon_{ij},$$

However, this model still has the **identifiability problem**:
The following transformation of the parameters does not change the model:

$$\alpha_i \to \alpha_i + c, \quad \beta_j \to \beta_j - c$$

for any constant c.
Additional constraints are needed to make the parameters unique.

Fixed Effect Model

We consider the following model:

$$X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

with $\sum_{i=1}^{I} \alpha_i = 0$ and $\sum_{j=1}^{J} \beta_j = 0$.

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The model is now **identifiable**.

Proof: Let $(\mu, \alpha_1, \ldots, \alpha_I, \beta_1, \ldots, \beta_J)$ and $(\mu', \alpha'_1, \ldots, \alpha'_I, \beta'_1, \ldots, \beta'_J)$ be two sets of parameters that give the same model. Then we have

$$\mu + \alpha_i + \beta_j = \mu' + \alpha'_i + \beta'_j \quad \text{for all } i, j.$$

Take the sum over i and j, we have $\mu = \mu'$. Take the sum over j, we have $\alpha_i = \alpha'_i$ for all i. Take the sum over i, we have $\beta_j = \beta'_j$ for all j.

Fixed Effect Model

$$X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

The interpretation of the parameters:

- μ: the grand mean
- α_i: the effect of the i-th level of factor A
- β_j: the effect of the j-th level of factor B

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Because:

- $\mu = E[\bar{X}_{..}]$
- $\alpha_i = E[\bar{X}_{i.}] - E[\bar{X}_{..}]$
- $\beta_j = E[\bar{X}_{.j}] - E[\bar{X}_{..}]$

Fixed Effect Model

$$X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

The parameters can be estimated unbiasedly by:

$$\hat{\mu} = \bar{X}_{..} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij}$$

$$\hat{\alpha}_i = \bar{X}_{i.} - \bar{X}_{..} = \frac{1}{J}\sum_{j=1}^{J} X_{ij} - \bar{X}_{..}$$

$$\hat{\beta}_j = \bar{X}_{.j} - \bar{X}_{..} = \frac{1}{I}\sum_{i=1}^{I} X_{ij} - \bar{X}_{..}$$

Verify that $\sum_{i=1}^{I} \hat{\alpha}_i = 0$ and $\sum_{j=1}^{J} \hat{\beta}_j = 0$.

Sum of Sqaures

We can define the following sum of squares for the two-factor ANOVA:

$$SST = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{..})^2 \qquad df: IJ - 1$$

$$SSA = J\sum_{i=1}^{I}(\bar{X}_{i.} - \bar{X}_{..})^2 \qquad df: I - 1$$

$$SSB = I\sum_{j=1}^{J}(\bar{X}_{.j} - \bar{X}_{..})^2 \qquad df: J - 1$$

$$SSE = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 \qquad df: (I-1)(J-1)$$

Verify that

$$SST = SSA + SSB + SSE$$

Mean Sqaures

We can define the following mean squares for the two-factor ANOVA:

$$MSA = \frac{SSA}{I-1}$$

$$MSB = \frac{SSB}{J-1}$$

$$MSE = \frac{SSE}{(I-1)(J-1)}$$

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The expected mean squares are:

$$E[MSA] = \sigma^2 + J \sum_{i=1}^{I} \alpha_i^2$$
$$E[MSB] = \sigma^2 + I \sum_{j=1}^{J} \beta_j^2$$
$$E[MSE] = \sigma^2$$

Hypothesis Testing

To test the main effects of factor A, we consider the following hypotheses:

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$$
$$H_a : \text{At least one } \alpha_i \text{ is not zero}$$

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Reject null when

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Similarly, to test the main effects of factor B, we consider the following hypotheses:

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_J = 0$$
$$H_a : \text{At least one } \beta_j \text{ is not zero}$$

Reject null when

$$F = \frac{MSB}{MSE} > F_{\alpha, J-1, (I-1)(J-1)}$$

From the Perspective of Nested Models

The testing of the main effect of factor A is equivalent to testing the following nested models:

Full model:	$X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$
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- The sum of squares error for the full model is SSE.
- The sum of squares error for the reduced model is $SSE + SSA$.

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- The sum of squares error for the full model is SSE.
- The sum of squares error for the reduced model is $SSE + SSA$.
- The F-test statistic is

$$F = \frac{(SSE + SSA - SSE)/(I-1)}{SSE/[(I-1)(J-1)]} = \frac{MSA}{MSE}$$

The ANOVA Table

The ANOVA table for the washing example is:

Source of Variation	df	Sum of Squares	Mean Square	f
Factor A (brand)	$I - 1 = 2$	SSA = .1282	MSA = .0641	$f_A = 4.43$
Factor B (wash treatment)	$J - 1 = 3$	SSB = .4797	MSB = .1599	$f_B = 11.05$
Error	$(I - 1)(J - 1) = 6$	SSE = .0868	MSE = .01447	
Total	$IJ - 1 = 11$	SST = .6947		

Tukey's Method for Multiple Comparison

The procedure is same as one-way ANOVA except that we have different thresholds for the different means:

$$w_A = Q_{\alpha,I,(I-1)(J-1)}\sqrt{MSE/J}, \quad w_B = Q_{\alpha,J,(I-1)(J-1)}\sqrt{MSE/I}$$

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- We use w_A to compare the means of factor A.
- We use w_B to compare the means of factor B.

Example

Recall the washing example.

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		1	2	3	4	Total	Average
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$$w = Q_{\alpha,4,6}\sqrt{MSE/3} = 0.34.$$

Therefore, the means of washing treatments 2, 3, 4 are close to each other.

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2. Randomly choose J units from the IJ units for the first level of factor A.
3. Randomly choose J units from the remaining $IJ - J$ units for the second level of factor A.
4. ...
5. Randomly choose J units from the remaining $2J$ units for the $(I - 1)$-th level of factor A.
6. The remaining J units are for the I-th level of factor A.

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4. Sample I units from the population in the second block.
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6. ...
7. Sample I units from the population in the J-th block.
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7. Sample I units from the population in the J-th block.
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The other way that random samples from each levels of factor A is randomly assigned to the blocks is also possible.

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- The humidity level is factor B with four levels.
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  - Sample $J = 4$ dehumidifiers from each brand.
  - Randomly assign the $J = 4$ dehumidifiers to the $J = 4$ humidity levels and test the power consumption.

Example

Treatments (brands)	Blocks (humidity level) 1	2	3	4	$x_{i\cdot}$	$\bar{x}_{i\cdot}$
1	685	792	838	875	3190	797.50
2	722	806	893	953	3374	843.50
3	733	802	880	941	3356	839.00
4	811	888	952	1005	3656	914.00
5	828	920	978	1023	3749	937.25
$x_{\cdot j}$	3779	4208	4541	4797	17,325	
$\bar{x}_{\cdot j}$	755.80	841.60	908.20	959.40		866.25

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Source of Variation	df	Sum of Squares	Mean Square	f
Treatments (brands)	4	53,231.00	13,307.75	$f_A = 95.57$
Blocks	3	116,217.75	38,739.25	$f_B = 278.20$
Error	12	1671.00	139.25	
Total	19	171,119.75		

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See also: **Simpson's paradox.**