STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 7: The Analysis of Variance II

Chencheng Cai

Washington State University

Example: One-way ANVOA

The fracture load of a certain type of material was measured under three different distances from the center.

						$x_{i\cdot}$
	42 mm:	2.62	2.99	3.39	2.86	11.86
Distance	36 mm:	3.47	3.85	3.77	3.63	14.72
	31.2 mm:	4.78	4.41	4.91	5.06	19.16
						$x_{\cdot\cdot} = 45.74$

- The factor: distance from the center.
- Three levels: 1, 2, 3.
- Each level has 4 observations.
- $x_{i\cdot}$ and $x_{\cdot\cdot}$ are the sums of observations.

Example: One-way ANVOA

- The mean observation at each level is

$$\bar{x}_{1\cdot} = x_{1\cdot}/4 = 2.965, \quad \bar{x}_{2\cdot} = x_{2\cdot}/4 = 3.680, \quad \bar{x}_{3\cdot} = x_{3\cdot}/4 = 4.790.$$

- The grand mean is

$$\bar{x}_{\cdot\cdot} = x_{\cdot\cdot}/12 = 3.812$$

Example: One-way ANVOA

- The sample variance within each level is

$$
\begin{aligned}
s_1^2 &= \frac{1}{4-1}\left[(2.62-2.965)^2+(2.99-2.965)^2+(3.39-2.965)^2+(2.86-2.965)^2\right] \\
&= 0.1038 \\
s_2^2 &= \frac{1}{4-1}\left[(3.47-3.680)^2+(3.85-3.680)^2+(3.77-3.680)^2+(3.63-3.680)^2\right] \\
&= 0.0279 \\
s_3^2 &= (4.78-4.790)^2+(4.41-4.790)^2+(4.91-4.790)^2+(5.06-4.790)^2 \\
&= 0.0773
\end{aligned}
$$

- The SSE is

$$SSE = (4-1)(s_1^2+s_2^2+s_3^2) = 0.6267$$

Example: One-way ANVOA

- The SSTr is

$$SSTr = 4\left[(2.965 - 3.812)^2 + (3.680 - 3.812)^2 + (4.790 - 3.812)^2\right]$$
$$= 6.7653$$

- The SST is

$$SST = SSTr + SSE = 7.3920$$

Example: One-way ANVOA

- The MSE is

$$MSE = \frac{SSE}{IJ - I} = \frac{0.6267}{12 - 3} = 0.0696$$

- The MSTr is

$$MSTr = \frac{SSTr}{I - 1} = \frac{6.7653}{3 - 1} = 3.3826$$

- The F-statistic is

$$F = \frac{MSTr}{MSE} = \frac{3.3826}{0.0696} = 48.60$$

- The p-value is

$$p = P(F_{2,8} > 48.60) < 0.0001$$

- The conclusion is that the distance from the center has a significant effect on the fracture load.

ANOVA Table

All the SS and MS information as well as F-statistic can be organized in an **ANOVA table**.

Source	df	SS	MS	F
Treatment	$I-1$	SSTr	MSTr	$F=\frac{MSTr}{MSE}$
Error	$IJ-I$	SSE	MSE	
Total	$IJ-1$	SST		

ANOVA Table

Source	df	SS	MS	F
Treatment	$I-1$	SSTr	MSTr	$F = \frac{MSTr}{MSE}$
Error	$IJ-I$	SSE	MSE	
Total	$IJ-1$	SST		

- "Treatment" can be replaced by the factor name, and "Error" can be replaced by "Residual".
- The **Total** variation is decomposed into **Treatment** and **Error** variation.
- Decomposition 1: Total df = Treatment df + Error df.
- Decomposition 2: Total SS = Treatment SS + Error SS.
- MS is computed per source of variation by dividing SS by df.
- The F-statistic is the ratio of MS for Treatment to MS for Error.
- The "Total" row is sometimes omitted.

Example

The ANOVA table for the fracture load example is

Source	df	SS	MS	F	p-value
Treatment	2	6.7653	3.3826	48.60	< 0.001
Error	9	0.6267	0.0696		
Total	11	7.3920			

Output from R:

```
             Df Sum Sq Mean Sq F value  Pr(>F)
level         2  6.765   3.383   48.58 1.5e-05 ***
Residuals     9  0.627   0.070
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Multiple Comparison

Suppose we want to compare the mean of one treatment level to another.

$$H_0 : \mu_i = \mu_{i'} \quad v.s. \quad H_a : \mu_i \neq \mu_{i'}$$

The two sample t-test (Pooled) gives the following CI for $\mu_i - \mu_{i'}$:

$$\bar{x}_i - \bar{x}_{i'} \pm t_{\alpha/2,2J-2}\sqrt{\frac{S_i^2 + S_{i'}^2}{2J}}$$

We only need check whether the CI contains 0.

- We have $I(I-1)/2$ pairs of comparisons.
- If all tests are independent with significance level α, the overall Type I error rate is $1-(1-\alpha)^{I(I-1)/2}$, whcih could be large when I is large.
- We need to adjust the significance level for each test to control the overall Type I error rate.

Multiple Comparison — Tukey's Method

The **simultaneous confidence interval** for $\mu_i - \mu_{i'}$ is

$$\bar{x}_i - \bar{x}_{i'} \pm Q_{\alpha,I,I(J-1)} \sqrt{\frac{MSE}{J}}$$

where $Q_{\alpha,I,I(J-1)}$ is the α-quantile of the **Studentized range distribution** with I and $I(J-1)$ degrees of freedom.

- There is at least $1-\alpha$ probability that the interval contains $\mu_i - \mu_{i'}$ for **every** pair of i and i'.
- The Studentized range distribution is a generalization of the Student's t-distribution for multiple comparisons.
- The quantile $Q_{\alpha,I,I(J-1)}$ can be found by `qtukey` in R.

Tukey's Procedure in Identifying Significant Differences

- Compute
$$w = Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}$$
- Order the treatment means from smallest to largest.
- Underscore maximal consecutive treatment means such that the difference of the maximum and minimum of the underscored means is less than w.

Tukey's Procedure in Identifying Significant Differences

Suppose we have $I = 5$ treatment levels with sample means:

$$\bar{x}_{1\cdot} = 14.5,\ \bar{x}_{2\cdot} = 13.8,\ \bar{x}_{3\cdot} = 13.3,\ \bar{x}_{4\cdot} = 14.3,\ \bar{x}_{5\cdot} = 13.1$$

We order the means:

$$\bar{x}_{5\cdot} = 13.1,\ \bar{x}_{3\cdot} = 13.3,\ \bar{x}_{2\cdot} = 13.8,\ \bar{x}_{4\cdot} = 14.3,\ \bar{x}_{1\cdot} = 14.5$$

Suppose we compute $w = 0.4$. Then we underscore the means:

$$\begin{array}{ccccc} \bar{x}_{5\cdot} & \bar{x}_{3\cdot} & \bar{x}_{2\cdot} & \bar{x}_{4\cdot} & \bar{x}_{1\cdot} \\ \underline{13.1 \quad 13.3} & & 13.8 & \underline{14.3 \quad 14.5} & \end{array}$$

Fixed Effect ANOVA Model

The anova model can be written as

$$X_{ij} = \mu_i + \epsilon_{ij}$$

where μ_i is the (unknown) mean of the ith treatment level and ϵ_{ij} is the random error term.

- The random error term is assumed to be normally distributed with mean 0 and variance σ^2.
- The random error term is independent of the treatment levels.
- The random error term is independent and identically distributed.
- The corresponding estimator of μ_i is $\hat{\mu}_i = \bar{X}_{i\cdot}$.

Fixed Effect ANOVA Model

We define

$$\mu = \frac{1}{I}\sum_{i=1}^{I}\mu_i, \qquad \alpha_i = \mu_i - \mu,$$

where

- μ is the (true) grand mean.
- α_i is the deviation of the ith treatment mean from the grand mean.
- α_i is also called the **effect** of the ith treatment level.

Then the model can be rewritten as

$$X_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

with the constraint

$$\sum_{i=1}^{I}\alpha_i = 0.$$

Fixed Effect ANOVA Model

$$X_{ij} = \mu + \alpha_i + \epsilon_{ij} \qquad \text{with } \sum_{i=1}^{I} \alpha_i = 0$$

- This is a **fixed effect model** because the treatment effects are fixed.
- The estimators are

$$\hat{\mu} = \bar{X}_{\cdot\cdot}, \quad \hat{\alpha}_i = \bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}.$$

- The ANOVA test is to test

$$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0.$$

Fixed Effect ANOVA Model

$$X_{ij} = \mu + \alpha_i + \epsilon_{ij} \qquad \text{with } \sum_{i=1}^{I} \alpha_i = 0$$

Our previous result shows, under the alternative,

$$E(MSE) = \sigma^2, \quad E(MSTr) \geq \sigma^2$$

We can have a more detailed result for the fixed effect ANOVA model:

$$E(MSTr) = \sigma^2 + \frac{J}{I-1} \sum_i \alpha_i^2.$$

Fixed Effect ANOVA Model

A quick proof:

$$SSTr = J\sum_i(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 = J\sum_i\left(\bar{X}_{i\cdot}^2 + \bar{X}_{\cdot\cdot}^2 - 2\bar{X}_{i\cdot}\bar{X}_{\cdot\cdot}\right) = J\left(\sum_i \bar{X}_{i\cdot}^2 - I\bar{X}_{\cdot\cdot}^2\right)$$

Because $\bar{X}_{i\cdot} \sim N(\mu_i, \sigma^2/J)$ and $\bar{X}_{\cdot\cdot} \sim N(\mu, \sigma^2/(IJ))$, we have

$$E(SSTr) = J\left(\sum_i E(\bar{X}_{i\cdot}^2) - I\cdot E(\bar{X}_{\cdot\cdot}^2)\right) = J\left(\sum_i\left(\mu_i^2 + \frac{\sigma^2}{J}\right) - I\left(\mu^2 + \frac{\sigma^2}{IJ}\right)\right)$$

$$= J\left(\sum_i \mu_i^2 - I\mu^2 + \frac{I-1}{J}\sigma^2 = J\sum_i \alpha_i^2 + (I-1)\sigma^2\right)$$

Therefore,

$$E(MSTr) = \frac{E(SSTr)}{I-1} = \sigma^2 + \frac{J}{I-1}\sum_i \alpha_i^2.$$

Random Effect ANOVA Model

We can also assume

$$X_{ij} = \mu + A_i + \epsilon_{ij}$$

where $A_i \sim N(0, \sigma_A^2)$ is the **random effect** of the ith treatment level.

- The treatment effect in a random effect model is a random variable.
- Although A_i is random, A_i remains constant for all observations in the ith treatment level.
- The random effect model is more flexible than the fixed effect model.
- The random effect model is more appropriate when the treatment levels are randomly selected from a larger population.

Random Effect ANOVA Model

If $A_1, A_2, \ldots, A_I$ are observed, we have the conditional mean of MSTr as

$$E(MSTr \mid A_1, A_2, \ldots, A_I) = \sigma^2 + \frac{J}{I-1} \sum_i (A_i - \bar{A})^2.$$

The unconditional mean of MSTr is

$$E(MSTr) = E\left[E(MSTr \mid A_1, A_2, \ldots, A_I)\right] = \sigma^2 + \frac{J}{I-1} E\left[\sum_i (A_i - \bar{A})^2\right] = \sigma^2 + J\sigma_A^2.$$

The hypothesis test becomes

$$H_0 : \sigma_A^2 = 0 \quad v.s. \quad H_a : \sigma_A^2 > 0.$$

Unequal Sample Sizes

Although we have assumed equal sample sizes in the ANOVA model for notational convinience, the ANOVA model can be extended to unequal sample sizes.

Now we assume J_i observations are taken at the ith treatment level and the total number of observations is N:

$$N = \sum_{i=1}^{I} J_i.$$

Unequal Sample Sizes

Sum of squares and mean squares are defined as before:

- SSE and MSE:

$$SSE = \sum_{i=1}^{I} \sum_{j=1}^{J_i} (X_{ij} - \bar{X}_{i\cdot})^2, \quad MSE = \frac{SSE}{N - I}.$$

- SSTr and MSTr:

$$SSTr = \sum_{i=1}^{I} J_i (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2, \quad MSTr = \frac{SSTr}{I - 1}.$$

The F-statistic is the same as before:

$$F = \frac{MSTr}{MSE}.$$

But its distribution under the null hypothesis is now $F_{I-1,N-I}$.

Unequal Sample Sizes

The Tukey's confidence interval for $\mu_i - \mu_{i'}$ should be adjusted to

$$\bar{x}_i - \bar{x}_{i'} \pm Q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}\left(\frac{1}{J_i} + \frac{1}{J_{i'}}\right)}.$$

Therefore, we need to compute the threshold difference as

$$w_{ii'} = Q_{\alpha,I,N-I}\sqrt{\frac{MSE}{2}\left(\frac{1}{J_i} + \frac{1}{J_{i'}}\right)}$$

for each pair of i and i'.

Example

The elastic modulus (GPa) obtained by a new ultrasonic method for specimens of a certain alloy produced using three different casting processes.

									J_i	$x_{i\cdot}$	$\bar{x}_{i\cdot}$
Permanent molding	45.5	45.3	45.4	44.4	44.6	43.9	44.6	44.0	8	357.7	44.71
Die casting	44.2	43.9	44.7	44.2	44.0	43.8	44.6	43.1	8	352.5	44.06
Plaster molding	46.0	45.9	44.8	46.2	45.1	45.5			6	273.5	45.58
									22	983.7	

Example

The anova table is

Source of Variation	df	Sum of Squares	Mean Square	f
Treatments	2	7.93	3.965	12.56
Error	19	6.00	.3158	
Total	21	13.93		

The p-value is

$$P(F_{2,19} > 5.52) = 0.013.$$

The conclusion is that the casting process has a significant effect on the elastic modulus.

Example

To compute Tukey's confidence interval for the difference of means is $(Q_{0.05,3,19} = 3.59)$

$$w_{12} = Q_{0.05,3,19}\sqrt{\frac{0.3158}{2}\left(\frac{1}{8}+\frac{1}{8}\right)} = 0.713$$

$$w_{13} = w_{23} = Q_{0.05,3,19}\sqrt{\frac{0.3158}{2}\left(\frac{1}{6}+\frac{1}{8}\right)} = 0.771$$

The conclusion is

2. Die	1. Permanent	3. Plaster
44.06	44.71	45.58