STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 6: The Analysis of Variance I

Chencheng Cai

Washington State University

Analysis of Variance

A **factor** is a qualitative variable that defines the groups to be compared.

The **levels** of a factor are the distinct values of the factor.

Examples: (factors highlighted)

- An experiment to study the effects of five different **brands** of gasoline on automobile engine operating efficiency (mpg).
- An experiment to study the effects of the presence of four different **sugar solutions** (glucose, sucrose, fructose, and a mixture of the three) on bacterial growth.
- An experiment to investigate whether **hardwood concentration in pulp** (%) at three different levels impacts tensile strength of bags made from the pulp.
- An experiment to decide whether the color density of fabric specimens depends on which of four different **dye amounts** is used

Analysis of Variance

Analysis of variance (ANOVA) is a statistical method used to compare the subpopulations of a factor.

- If there is one factor, it is called **one-way ANOVA** or **single-factor ANOVA**.
- If there is one factor with two levels, the ANOVA should be similar to a two-sample test.
- All examples in the previous slide are one-way ANOVA.
- If there are two (or more) factors, it is called **two-way ANOVA** (or **multi-factor ANOVA**).
- Example of two-way ANOVA:
  An experiment to study the effects of two factors, **temperature** and **humidity**, on the growth of a certain type of bacteria.

One-way ANOVA — Notations

- I: the number of levels of the factor.
- $\mu_i, i = 1, \ldots, I$: the population mean of the ith level of the factor.
- The relevant hypotheses:

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I$$
$$H_a : \text{At least one of the means is different}$$

- In experimental design, the ith level of the factor is often called a **treatment**.
- X_{ij} is the jth observation in the ith treatment.
- x_{ij} is the value of X_{ij} when the experiment is conducted.

Example

Compress strength of different types of boxes.

Type of Box	Compression Strength (lb)						Sample Mean	Sample SD
1	655.5	788.3	734.3	721.4	679.1	699.4	713.00	46.55
2	789.2	772.5	786.9	686.1	732.1	774.8	756.93	40.34
3	737.1	639.0	696.3	671.7	717.2	727.1	698.07	37.20
4	535.1	628.7	542.4	559.0	586.9	520.0	562.02	39.87
						Grand mean =	682.50	

Different Means

Let X_{ij} be the j-th observation in the i-th treatment.

Suppose each treatment level has J observations. Then the total number of observations is $I \times J$.

- The sample mean of the i-th treatment is

$$\bar{X}_{i\cdot} = \frac{1}{J} \sum_{j=1}^{J} X_{ij}$$

- The sample mean of all observations (**grand mean**) is

$$\bar{X}_{\cdot\cdot} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} X_{ij}$$

- Q: what if we have unequal number of observations in each treatment?

Sum of Squares

- The **Sum of Squares Error** (SSE) is

$$SSE = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{i\cdot})^2$$

- The **Sum of Squares Treatment** (SSTr) is

$$SSTr = J \sum_{i=1}^{I} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$$

- The **Sum of Squares Total** (SST or SSTo) is

$$SST = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{\cdot\cdot})^2$$

Sum of Squares

The relationship between SST, SSTr, and SSE:

$$SST = SSTr + SSE$$

Q: what if we have unequal number of observations in each treatment?

Mean Sqaures

The mean sqaures are the sum of squares divided by the degrees of freedom.

$$MSX = \frac{SSX}{\text{degrees of freedom}}$$

The degrees of freedom (df) can be calculated as

$$\text{df} = \text{number of observations} - \text{number of parameters}$$

Mean Sqaures

- The **Mean Square Error** (MSE) is

$$MSE = \frac{SSE}{IJ - I} = \frac{1}{IJ - I} \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{i\cdot})^2$$

- The **Mean Square Treatment** (MSTr) is

$$MSTr = \frac{SSTr}{I - 1} = \frac{J}{I - 1} \sum_{i=1}^{I} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$$

Nested Models

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I \quad \text{v.s.} \quad H_a : \text{not all equal}$$

- The **full model** is the model with all the treatment means different. (i.e. $H_0 \cup H_a$)
  The estimators are
  $$\hat{\mu}_i = \bar{X}_{i\cdot} \quad \text{for } i = 1, \ldots, I.$$
- The **reduced model** is the model with all the treatment means equal. (i.e. H_0)
  The estimator is
  $$\hat{\mu}_i = \hat{\mu} = \bar{X}_{\cdot\cdot} \quad \text{for } i = 1, \ldots, I.$$
- The two models are **nested** because the reduced model is a special case of the full model.

Nested Models

The sum of squared error for the full model is

$$SSE_{full} = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \hat{\mu}_i)^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{i\cdot})^2 = SSE$$

The sum of squared error for the reduced model is

$$SSE_{reduced} = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \hat{\mu}_i)^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{\cdot\cdot})^2 = SST$$

- The **extra sum of squares** is

$$SSE_{reduced} - SSE_{full} = SST - SSE = SSTr$$

- The full model uses $I - 1$ more parameters than the reduced model.
- The full model improves the fit by $SSTr$.

Nested Models

For the full model:

- Sum of squared error is SSE with $IJ - I$ degrees of freedom.
- Sum of squares fitted is $SSTr$ with $I - 1$ degrees of freedom.
- Sum of squares total is SST with $IJ - 1$ degrees of freedom.

For the reduced model:

- Sum of squared error is SST with $IJ - 1$ degrees of freedom.
- Sum of squares fitted is 0 with 0 degrees of freedom.
- Sum of squares total is SST with $IJ - 1$ degrees of freedom.

F-Test for Nested Models

In order to test the nested model hypothesis:

$$H_0 : \text{reduced model is true} \quad \text{v.s.} \quad H_a : \text{full model is true}$$

We consider the following F-statistic:

$$F = \frac{(SSE_{reduced} - SSE_{full})/\text{difference in d.f.}}{SSE_{full}/\text{residual d.f. of full model}}$$

Large F-statistic suggests that the increase in fit is significant by considering the full model.

We reject null hypothesis if F is large enough.

F-Test for One-way ANOVA

In order to test the hypothesis

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I \quad \text{v.s.} \quad H_a : \text{not all equal}$$

We consider the following F-statistic:

$$F = \frac{MSTr}{MSE} = \frac{SSTr/(I-1)}{SSE/(IJ-I)}$$

Intuition:

- The numerator measures the variability between the treatment means.
- The denominator measures the variability within the treatments.
- A large F-statistic suggests that the treatment means are different.

F-Test for One-way ANOVA

Under the following assumptions:

- The observations are independent.
- The populations are normally distributed.
- The populations have the same variance.

The F-statistic follows an F-distribution with $I-1$ and $IJ-I$ degrees of freedom, denoted by $F_{I-1,IJ-I}$.

The decision rule is

- Reject H_0 if $F > F_{\alpha,I-1,IJ-I}$.
- Fail to reject H_0 if $F \leq F_{\alpha,I-1,IJ-I}$.
- The p-value is $P(F > F_{obs})$.
- The critical value is $F_{\alpha,I-1,IJ-I}$.

Background — Chi Square (χ^2) Distribution

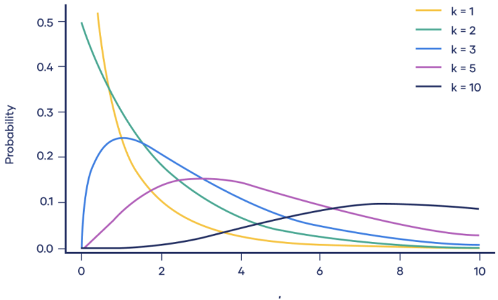
- If $Z_1, Z_2, \ldots, Z_k$ are independent standard normal random variables, then the sum of their squares

$$Q = Z_1^2 + Z_2^2 + \cdots + Z_k^2$$

follows a χ^2 distribution with k degrees of freedom, denoted by χ_k^2.

- The χ^2 distribution is supported on $[0, \infty)$.
- Mean and variance:

$$E(Q) = k, \quad \mathrm{Var}(Q) = 2k$$



Background — Chi Square (χ^2) Distribution

- If $Q_1 \sim \chi^2_{k_1}$ and $Q_2 \sim \chi^2_{k_2}$ are independent, then

$$Q = Q_1 + Q_2 \sim \chi^2_{k_1+k_2}$$

- If $X_1, X_2, \ldots, X_k \sim N(0,1)$, then

$$\sum_{i=1}^{k} (X_i - \bar{X})^2 \sim \chi^2_{k-1} \quad \text{with} \quad \bar{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$$

- If $X_1, X_2, \ldots, X_k \sim N(\mu,1)$, then

$$\sum_{i=1}^{k} (X_i - \bar{X})^2 \sim \chi^2_{k-1}.$$

- If $X_1, X_2, \ldots, X_k \sim N(\mu,\sigma^2)$, then

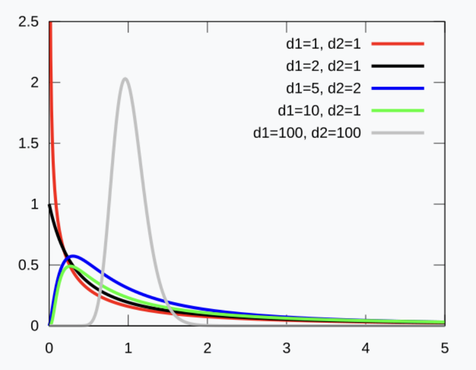
$$\sum_{i=1}^{k} (X_i - \bar{X})^2 \sim \sigma^2 \cdot \chi^2_{k-1}.$$

Background — F Distribution

- If $Q_1 \sim \chi^2_{k_1}$ and $Q_2 \sim \chi^2_{k_2}$ are independent, then

$$F = \frac{Q_1/k_1}{Q_2/k_2}$$

  follows an F-distribution with k_1 and k_2 degrees of freedom, denoted by F_{k_1,k_2}.
- The F-distribution is supported on $[0, \infty)$.



F Distribution in ANOVA

Our assumption is that $X_{ij} \sim N(\mu_i, \sigma^2)$ for all i and j.

One the ond hand,

$$\sum_{j=1}^{J} (X_{ij} - \bar{X}_{i\cdot})^2 \sim \sigma^2 \cdot \chi^2_{J-1}$$

Therefore,

$$SSE = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{i\cdot})^2 \sim \sigma^2 \chi^2_{IJ-I}$$

F Distribution in ANOVA

Our assumption is that $X_{ij} \sim N(\mu_i, \sigma^2)$ for all i and j.

On the other hand,

$$\bar{X}_{i\cdot} \sim N(\mu_i, \sigma^2/J).$$

Therefore, under null hypothesis ($\mu_i = \mu$ for all i),

$$\sum_{i=1}^{I} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 \sim \frac{\sigma^2}{J} \cdot \chi^2_{I-1}.$$

Then

$$SSTr = J \sum_{i=1}^{I} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 \sim \sigma^2 \cdot \chi^2_{I-1}.$$

F Distribution in ANOVA

Recall our previous results:

- $SSE \sim \sigma^2 \cdot \chi^2_{IJ-I}$.
- $SSTr \sim \sigma^2 \cdot \chi^2_{I-1}$ under null hypothesis.

Given that SSE and $SSTr$ are independent (beyond the scope), we have, under null hypothesis,

$$F = \frac{MSTr}{MSE} = \frac{SSTr/(I-1)}{SSE/(IJ-I)} \sim \frac{\chi^2_{I-1}/(I-1)}{\chi^2_{IJ-I}/(IJ-I)} \sim F_{I-1,IJ-I}.$$

F Distribution in ANOVA

Under null hypothesis,

- Because $SSE \sim \sigma^2 \cdot \chi^2_{IJ-I}$, we have

$$E(SSE) = (IJ - I)\sigma^2, \quad E(MSE) = \sigma^2.$$

- Because $SSTr \sim \sigma^2 \cdot \chi^2_{I-1}$, we have

$$E(SSTr) = (I - 1)\sigma^2, \quad E(MSTr) = \sigma^2.$$

Under alternative hypothesis,

- We still have

$$E(SSE) = (IJ - I)\sigma^2, \quad E(MSE) = \sigma^2.$$

- But for $SSTr$, we have

$$E(SSTr) > (I - 1)\sigma^2, \quad E(MSTr) > \sigma^2.$$