STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 6: The Analysis of Variance

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- An experiment to study the effects of the presence of four different **sugar solutions** (glucose, sucrose, fructose, and a mixture of the three) on bacterial growth.

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- An experiment to investigate whether **hardwood concentration in pulp** (%) at three different levels impacts tensile strength of bags made from the pulp.
- An experiment to decide whether the color density of fabric specimens depends on which of four different **dye amounts** is used

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- All examples in the previous slide are one-way ANOVA.
- If there are two (or more) factors, it is called **two-way ANOVA** (or **multi-factor ANOVA**).
- Example of two-way ANOVA:
  An experiment to study the effects of two factors, **temperature** and **humidity**, on the growth of a certain type of bacteria.

One-way ANOVA — Notations

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- X_{ij} is the jth observation in the ith treatment.
- x_{ij} is the value of X_{ij} when the experiment is conducted.

Example

Compress strength of different types of boxes.

Type of Box	Compression Strength (lb)						Sample Mean	Sample SD
1	655.5	788.3	734.3	721.4	679.1	699.4	713.00	46.55
2	789.2	772.5	786.9	686.1	732.1	774.8	756.93	40.34
3	737.1	639.0	696.3	671.7	717.2	727.1	698.07	37.20
4	535.1	628.7	542.4	559.0	586.9	520.0	562.02	39.87
					Grand mean =		682.50	

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Sum of Squares

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- The **Sum of Squares Total** (SST or SSTo) is

$$SST = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{\cdot\cdot})^2$$

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The degrees of freedom (df) can be calculated as

$$\text{df} = \text{number of observations} - \text{number of parameters}$$

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- The **Mean Square Treatment** (MSTr) is

$$MSTr = \frac{SSTr}{I - 1} = \frac{J}{I - 1} \sum_{i=1}^{I} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2$$

Nested Models

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I \quad \text{v.s.} \quad H_a : \text{not all equal}$$

- The **full model** is the model with all the treatment means different. (i.e. $H_0 \cup H_a$)
  The estimators are
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  $$\hat{\mu}_i = \hat{\mu} = \bar{X}_{\cdot\cdot} \quad \text{for } i = 1, \ldots, I.$$
- The two models are **nested** because the reduced model is a special case of the full model.

Nested Models

The sum of squared error for the full model is

$$SSE_{full} = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \hat{\mu}_i)^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_{i\cdot})^2 = SSE$$

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- The full model uses $I - 1$ more parameters than the reduced model.
- The full model improves the fit by $SSTr$.

Nested Models

For the full model:

- Sum of squared error is SSE with $IJ - I$ degrees of freedom.
- Sum of squares fitted is $SSTr$ with $I - 1$ degrees of freedom.
- Sum of squares total is SST with $IJ - 1$ degrees of freedom.

For the reduced model:

- Sum of squared error is SST with $IJ - 1$ degrees of freedom.
- Sum of squares fitted is 0 with 0 degrees of freedom.
- Sum of squares total is SST with $IJ - 1$ degrees of freedom.

F-Test for Nested Models

In order to test the nested model hypothesis:

$$H_0 : \text{reduced model is true} \quad \text{v.s.} \quad H_a : \text{full model is true}$$

We consider the following F-statistic:

$$F = \frac{(SSE_{reduced} - SSE_{full})/\text{difference in d.f.}}{SSE_{full}/\text{residual d.f. of full model}}$$

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We reject null hypothesis if F is large enough.

F-Test for One-way ANOVA

In order to test the hypothesis

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I \quad \text{v.s.} \quad H_a : \text{not all equal}$$

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Intuition:

- The numerator measures the variability between the treatment means.
- The denominator measures the variability within the treatments.
- A large F-statistic suggests that the treatment means are different.

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The decision rule is

- Reject H_0 if $F > F_{\alpha,I-1,IJ-I}$.
- Fail to reject H_0 if $F \leq F_{\alpha,I-1,IJ-I}$.
- The p-value is $P(F > F_{obs})$.
- The critical value is $F_{\alpha,I-1,IJ-I}$.

Background — Chi Square (χ^2) Distribution

- If $Z_1, Z_2, \ldots, Z_k$ are independent standard normal random variables, then the sum of their squares

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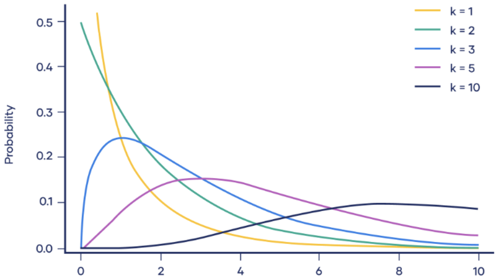

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$$\sum_{i=1}^{k} (X_i - \bar{X})^2 \sim \chi^2_{k-1} \quad \text{with} \quad \bar{X} = \frac{1}{k} \sum_{i=1}^{k} X_i$$

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- If $X_1, X_2, \ldots, X_k \sim N(\mu,\sigma^2)$, then

$$\sum_{i=1}^{k} (X_i - \bar{X})^2 \sim \sigma^2 \cdot \chi^2_{k-1}.$$

Background — F Distribution

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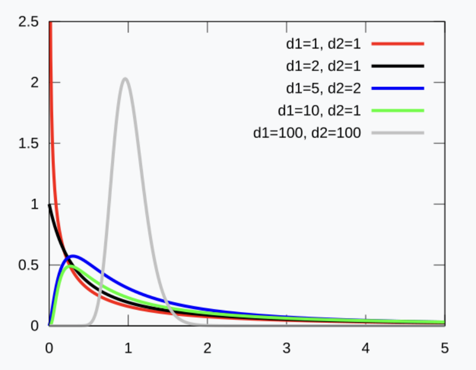

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$$\sum_{j=1}^{J} (X_{ij} - \bar{X}_{i\cdot})^2 \sim \sigma^2 \cdot \chi^2_{J-1}$$

Therefore,

$$SSE = \sum_{i=1}^{I} \sum_{j=1}^{J} (X_{ij} - \bar{X}_{i\cdot})^2 \sim \sigma^2 \chi^2_{IJ-I}$$

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On the other hand,

$$\bar{X}_{i\cdot} \sim N(\mu_i, \sigma^2/J).$$

Therefore, under null hypothesis ($\mu_i = \mu$ for all i),

$$\sum_{i=1}^{I} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 \sim \frac{\sigma^2}{J} \cdot \chi^2_{I-1}.$$

Then

$$SSTr = J \sum_{i=1}^{I} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 \sim \sigma^2 \cdot \chi^2_{I-1}.$$

F Distribution in ANOVA

Recall our previous results:

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- $SSTr \sim \sigma^2 \cdot \chi^2_{I-1}$ under null hypothesis.

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- $SSTr \sim \sigma^2 \cdot \chi^2_{I-1}$ under null hypothesis.

Given that SSE and $SSTr$ are independent (beyond the scope), we have, under null hypothesis,

$$F = \frac{MSTr}{MSE} = \frac{SSTr/(I-1)}{SSE/(IJ-I)} \sim \frac{\chi^2_{I-1}/(I-1)}{\chi^2_{IJ-I}/(IJ-I)} \sim F_{I-1,IJ-I}.$$

F Distribution in ANOVA

Under null hypothesis,

- Because $SSE \sim \sigma^2 \cdot \chi^2_{IJ-I}$, we have

$$E(SSE) = (IJ - I)\sigma^2, \quad E(MSE) = \sigma^2.$$

- Because $SSTr \sim \sigma^2 \cdot \chi^2_{I-1}$, we have

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Under alternative hypothesis,

- We still have

$$E(SSE) = (IJ - I)\sigma^2, \quad E(MSE) = \sigma^2.$$

- But for $SSTr$, we have

$$E(SSTr) > (I - 1)\sigma^2, \quad E(MSTr) > \sigma^2.$$