STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 3: Point Estimation II

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Methods of Point Estimation

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Now we introduce some methods to construct point estimators:

- ▶ Method of Moments (MoM)
- ▶ Maximum Likelihood Estimation (MLE)

Method of Moments (MoM)

Definition (Moments)

Let $X_1, \ldots, X_n$ be a random sample from a population with pmf or pdf $f(x)$. For $k = 1, 2, \ldots$, the **kth population moment** or **kth moment of the distribution** $f(x)$, is $E(X^k)$. The **kth sample moment** is

$$\frac{1}{n} \sum_{i=1}^{n} X_i^k.$$

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$$\frac{1}{n} \sum_{i=1}^{n} X_i^k.$$

- ▶ kth moment of a distribution is the expected value of X^k.
- ▶ kth sample moment is the sample average of X^k.
- ▶ When $n \to \infty$, the two moments are equal (by Law of Large Numbers).

Method of Moments (MoM)

Let $X_1, \ldots, X_n$ be a random sample from a population with pmf or pdf $f(x; \theta_1, \ldots, \theta_m)$, where $\theta_1, \ldots, \theta_m$ are the unknown parameters we want to estimate.

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The **method of moments estimator** of $\theta_1, \ldots, \theta_m$ is the solution to the following system of equations:

$$\frac{1}{n} \sum_{i=1}^{n} X_i = E(X) = g_1(\theta_1, \ldots, \theta_m)$$

$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 = E(X^2) = g_2(\theta_1, \ldots, \theta_m)$$

$$\vdots$$

$$\frac{1}{n} \sum_{i=1}^{n} X_i^m = E(X^k) = g_m(\theta_1, \ldots, \theta_m)$$

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$$\frac{1}{n} \sum_{i=1}^{n} X_i^m = E(X^k) = g_m(\theta_1, \ldots, \theta_m)$$

In short: **MoM matches the sample moments with the population moments**.

Example

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MoM matches the first two moments:

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$$\frac{1}{n} \sum_{i=1}^{n} X_i^2 = E(X^2) = E(X)^2 + \mathrm{Var}(X) = \mu^2 + \sigma^2$$

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The solution, the MoM estimator, is

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \overline{X^2} - (\bar{X})^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Example (Textbook 6.13)

Let $X_1, \ldots, X_n$ be a random sample from a Gamma distribution with parameters α and β. The pdf is

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}.$$

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$$E(X) = \alpha\beta, \quad E(X^2) = \alpha(\alpha+1)\beta^2.$$

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The MoM estimator of α and β are the solutions to the following equations:

$$\frac{1}{n}\sum_{i=1}^{n} X_i = E(X) = \alpha\beta$$

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The solutions are

$$\hat{\alpha} = \frac{\bar{X}^2}{\overline{X^2} - \bar{X}^2}$$

$$\hat{\beta} = \frac{\overline{X^2} - \bar{X}^2}{\bar{X}}$$

Method of Moments

- MoM only requires the first few moments of the distribution. (Not the explicit pmf or pdf)
- If the first m moments do not give a unique solution, we can use more moments.
- MoM estimator is approximately normal if the sample size is large enough (by CLT).

Maximum Likelihood Estimation (MLE)

Definition (Likelihood Function)

Let $X_1, \ldots, X_n$ be a random sample from a population with pmf or pdf $f(x; \theta_1, \ldots, \theta_m)$. The **likelihood function** is

$$L(\theta_1, \ldots, \theta_m) = \prod_{i=1}^{n} f(x_i; \theta_1, \ldots, \theta_m).$$

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$$L(\theta_1, \ldots, \theta_m) = \prod_{i=1}^{n} f(x_i; \theta_1, \ldots, \theta_m).$$

- The likelihood function is a function of the parameters $\theta_1, \ldots, \theta_m$.
- though it has exactly the same formula as the joint pmf or pdf of the sample.
- In order to compute the likelihood, we need to know the pmf or pdf explicitly. (compare it to MoM)

Important Clarificaitons on Likelihood

- ▶ The likelihood function is **the probability of observing the sample** given the parameters.
- ▶ **NOT** the probability of the parameters given the sample.

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▶ **NOT** the probability of the parameters given the sample.

▶ That is

$$L(\theta_1, \ldots, \theta_m) \neq p(\theta_1, \ldots, \theta_m \mid X_1, \ldots, X_n)$$

▶ The r.h.s. of above is

$$p(\theta_1, \ldots, \theta_m \mid X_1, \ldots, X_n) = \frac{p(X_1, \ldots, X_n \mid \theta_1, \ldots, \theta_m)p(\theta_1, \ldots, \theta_m)}{p(X_1, \ldots, X_n)}$$

but we assume $\theta_1, \ldots, \theta_m$ to be fixed.

Example (Textbook 6.15)

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The likelihood function is

$$L(p) = f(x_1, \ldots, x_{10}; p) = p(1-p)p(1-p)\cdots p = p^3(1-p)^7.$$

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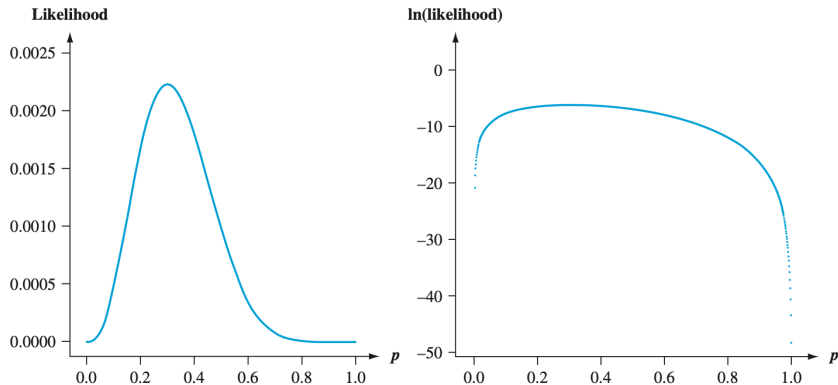
The likelihood function is

$$L(p) = f(x_1, \ldots, x_{10}; p) = p(1-p)p(1-p) \cdots p = p^3(1-p)^7.$$

The logarithm of the likelihood function is called the **log-likelihood function**:

$$\ell(p) := \log L(p) = 3 \log p + 7 \log(1-p).$$

Example (Textbook 6.15)

The intuitively best guess of p is the value that maximizes the likelihood function.

Maximum Likelihood Estimation (MLE)

The **maximum likelihood estimator** of $\theta_1, \ldots, \theta_m$ is the value of $\theta_1, \ldots, \theta_m$ that maximizes the likelihood function $L(\theta_1, \ldots, \theta_m)$.

The **log-likelihood function** is

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In many cases, the MLE of $\theta_1, \ldots, \theta_m$ is the solution to the following system of equations: The MLE of $\theta_1, \ldots, \theta_m$ is the solution to the following system of equations:

$$\frac{\partial \ell}{\partial \theta_1} = 0, \ldots, \frac{\partial \ell}{\partial \theta_m} = 0.$$

The first order derivatives are called the **score functions**. The MLE is a zero of the score functions.

Example (Textbook 6.15) Cont.

Continue the example of passwords. The score function is

$$\frac{d\ell(p)}{dp} = \frac{d(3\log p + 7\log(1-p))}{dp} = \frac{3}{p} - \frac{7}{1-p}.$$

Continue the example of passwords. The score function is

$$\frac{d\ell(p)}{dp} = \frac{d(3\log p + 7\log(1-p))}{dp} = \frac{3}{p} - \frac{7}{1-p}.$$

The MLE is the solution to

$$\frac{3}{p} - \frac{7}{1-p} = 0.$$

The solution is $\hat{p} = 3/10$.

Example

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The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} X_i}.$$

The log-likelihood function is

$$\ell(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} X_i.$$

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The log-likelihood function is

$$\ell(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} X_i.$$

The score function is

$$\frac{d\ell(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i.$$

Example

The MLE is the solution to

$$\frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0.$$

The solution is $\hat{\lambda} = n / \sum_{i=1}^{n} X_i = 1/\bar{X}$.

Example

The MLE is the solution to

$$\frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0.$$

The solution is $\hat{\lambda} = n/\sum_{i=1}^{n} X_i = 1/\bar{X}$.

► $\hat{\lambda}$ is biased because (Jensen's inequality)

$$E\left(\frac{1}{\bar{X}}\right) > \frac{1}{E(\bar{X})} = \lambda.$$

► The MoM estimator for λ is $\hat{\lambda} = 1/\bar{X}$ as well (based on the first moment).

Example

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The likelihood function is

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_i - \mu)^2}{2\sigma^2}\right) = \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} \exp\left(-\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{2\sigma^2}\right)$$

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The log-likelihood function is

$$\ell(\mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2.$$

The score functions are

$$\frac{\partial\ell}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu), \quad \frac{\partial\ell}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(X_i - \mu)^2.$$

Example

The MLE is the solution to

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu) = 0, \quad -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^{n} (X_i - \mu)^2 = 0.$$

The solution is

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

The MLE of μ is the sample mean, and the MLE of σ^2 is the sample variance.

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The MLE of μ is the sample mean, and the MLE of σ^2 is the sample variance.

▶ $\hat{\mu}$ is unbiased.
▶ $\hat{\sigma}^2$ is biased.

Maximum Likelihood Estimation

- ▶ MLE is not unique. (The likelihood function may have multiple maxima.)
- ▶ MLE is not always the solution to the score functions. (The score functions may not have zeros.)
- ▶ When the sample size is large enough and the MLE is a zero of the score functions, the MLE is approximately normal (by CLT).
- ▶ When the sample size is large enough, the MLE is approximately unbiased (by Law of Large Numbers).
- ▶ When the sample size is large enough, the MLE is approximately efficient (with smallest variance).
- ▶ MLE is transformation invariant. (If $\hat{\theta}$ is the MLE of θ, then $\phi(\hat{\theta})$ is the MLE of $\phi(\theta)$ for any function ϕ).

Example: Domain-related Distribution

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The likelihood function is

$$L(\theta) = \frac{1}{\theta^n} I(\max(X_1, \ldots, X_n) \leq \theta).$$

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The likelihood function is monotone decreasing in $\theta > X_{max}$. The MLE is $\hat{\theta} = X_{max}$.

- $\hat{\theta}$ is biased. (because $E(X_{max}) < \theta$)
- When the sample size is large enough, $\hat{\theta}$ is **not** approximately normal.

Example: Domain-related Distribution

If we consider MoM for the same problem. The MoM estimator is the solution to

$$\frac{1}{n} \sum_{i=1}^{n} X_i = E(X) = \frac{\theta}{2}.$$

The MoM estimator is $\hat{\theta} = 2\bar{X}$.

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- $\hat{\theta}$ is unbiased.
- When the sample size is large enough, $\hat{\theta}$ is approximately normal.
- However, it could happen that $\hat{\theta} < X_{max}$.

Beyond Point Estimation

▶ The point estimation gives a single number as the estimate of the parameter.

▶ The performance of the point estimator is measured by the bias, variance, and mean squared error.

▶ However, the bias, variance and mean squared error could depend on the true value of the parameter, and are therefore not always informative.

▶ The point estimator lacks a probabilistic statement about the uncertainty of the estimate.

Confidence Interval

A **confidence interval (CI)** for an univariate parameter θ is an interval $[L, U]$ such that

$$P(L \leq \theta \leq U) = 1 - \alpha,$$

where α is the **significance level** and $1 - \alpha$ is the **confidence level**.

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where α is the **significance level** and $1 - \alpha$ is the **confidence level**.

- ▶ Given confidence level $1 - \alpha$, the confidence interval is not unique.
- ▶ When there are multiple parameters, the confidence interval becomes a confidence region.
- ▶ The interval can be closed, open or half-open.
- ▶ A common value for α is 0.05, which corresponds to the 95% confidence level.
- ▶ A conservative confidence interval satisfies

$$P(L \leq \theta \leq U) \geq 1 - \alpha.$$

It usually happens for discrete parameters.

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Interpretation:

- The confidence interval $[L, U]$ is a random interval that contains the true parameter θ with probability $1 - \alpha$.
- If we repeat the experiment many times, in expectation, $(1 - \alpha) \times 100\%$ of the confidence intervals will contain the true parameter.

Confidence Interval Interpretation
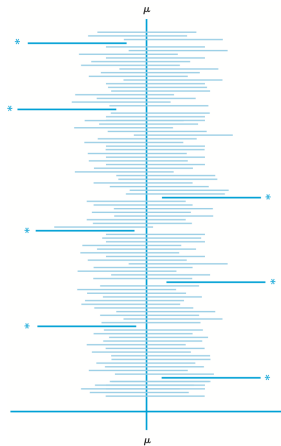
Confidence Interval for the Population Mean

Proposition

Let $X_1, \ldots, X_n$ be a random sample from a **normal distributed** *population with mean μ and variance σ^2. The 95% confidence interval for μ is*

$$\left(\bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \quad \bar{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} \right).$$

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$$\left(\bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \quad \bar{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} \right).$$

- ▶ The 1.96 is the 97.5 percentile of the standard normal distribution.
- ▶ The 95% confidence interval is symmetric around the sample mean $\bar{X}$.
- ▶ The confidence interval is exact. (No approximation is involved.)
- ▶ The confidence interval requires that σ is known.

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We now verify the proposition.

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▶ Step 2: normalize the event:

$$\left\{ \bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} \right\} = \left\{ -1.96 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 1.96 \right\},$$

we can define $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$.

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we can define $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$.

▶ Step 3: from Step 1, we know $Z \sim N(0, 1)$. Therefore,

$$P(-1.96 < Z < 1.96) = \Phi(1.96) - \Phi(-1.96) = 0.95,$$

where Φ is the cdf of the standard normal distribution.

Generalization

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- By CLT, for large n, $\bar{X}$ is approximately normal.

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Proposition

Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2.
The **approximate** 95% *confidence interval for μ is*

$$\left(\bar{X} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \quad \bar{X} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} \right).$$

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whose distribution does not depend on the parameters.

We call such transformation of the data the **pivot** or **standardized statistic**.

The pivot is extremely useful in constructing confidence intervals.

Generalization

In Step 3, we find the quantiles of the pivot to ensure the coverage probability: that is, -1.96 and 1.96 are selected such that

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In Step 3, we find the quantiles of the pivot to ensure the coverage probability: that is, -1.96 and 1.96 are selected such that

$$P(-1.96 < Z < 1.96) = 0.95.$$

▶ For an arbitrary confidence level $1 - \alpha$, we need to find $z_{\alpha/2}$, which is the quantiles for the standard normal:

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha.$$

▶ The $1 - \alpha$ confidence interval is not necessarily symmetric. That is

$$P(-z_{\beta} < Z < z_{\alpha-\beta}) = 1 - \alpha,$$

for any choice of $\beta \in (0, \alpha)$.
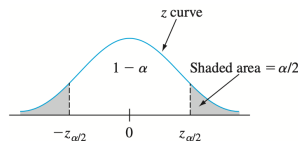
▶ But the symmetric one has the shortest length.

Figure 7.4 $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$

Confidnece Interval for Population Mean (Generalized)

Proposition

Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2.
The **approximate** $100(1 - \alpha)\%$ confidence interval for μ is

$$\left(\bar{X} - z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}}, \quad \bar{X} + z_{\alpha/2} \cdot \frac{\hat{\sigma}}{\sqrt{n}} \right),$$

where

$$\hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$$

is the sample standard deviation, and $z_{\alpha/2}$ is the $\alpha/2$ quantile of the standard normal distribution.

Pivot Trick Example

Let $X_1, \ldots, X_n$ be a random sample from a population distributed as $N(0, \sigma^2)$.
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- Notice that $\bar{X} \sim N(0, \sigma^2/n)$.
- Therefore,
$$Z = \frac{\bar{X}}{\sigma/\sqrt{n}} \sim |N(0,1)|,$$

is a pivot.

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▶ Therefore,

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is a pivot.

▶ The 95% confidence interval for σ is

$$\{0 < Z < 1.96\} = \left\{ \frac{\sqrt{n}\bar{X}}{1.96} < \sigma < \infty \right\}.$$

The CLT Trick Example

The CLT is used to give approximate CI for the population mean without a normal assumption.

Let $X_1, \ldots, X_n$ be Bernoulli random variables with success probability p.

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Let $X_1, \ldots, X_n$ be Bernoulli random variables with success probability p.

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$$\bar{X} \approx N(p, p(1-p)/n).$$

▶ The 95% confidence interval for p is

$$\left(\bar{X} - 1.96 \cdot \sqrt{\frac{\bar{X}(1-\bar{X})}{n}}, \bar{X} + 1.96 \cdot \sqrt{\frac{\bar{X}(1-\bar{X})}{n}} \right).$$

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▶ Remark: Textbook uses a more refined CI for the success probability, which is more complicated.

Exact Confidence Interval for Normal

- In our CI for the normal population mean, we assume that the population variance σ^2 is known.
- If σ^2 is unknown, we use the sample variance $\hat{\sigma}^2$ instead. This is called the **plug-in** method.

Exact Confidence Interval for Normal

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▶ If σ^2 is unknown, we use the sample variance $\hat{\sigma}^2$ instead. This is called the **plug-in** method.

▶ The plug-in method is not exact. The CI is only approximately valid.

▶ In order to construct an exact CI for normal population mean, we need to investigate the **exact** distribution of

$$\frac{\bar{X} - \mu}{S/\sqrt{n}},$$

where S is the sample standard deviation.

t-distribution

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. The random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

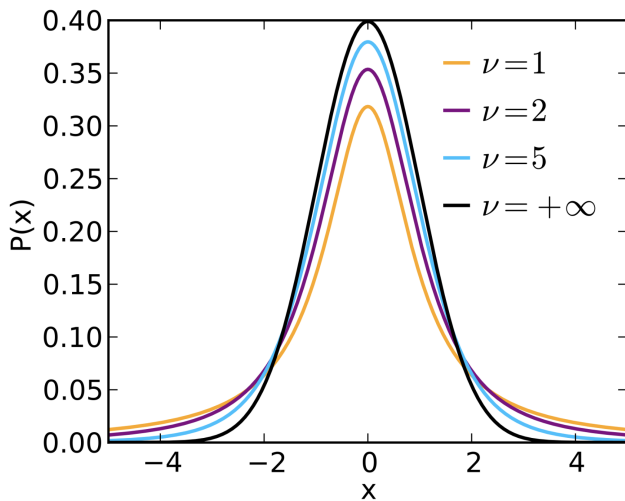follows a **t-distribution** with $n - 1$ degrees of freedom.

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follows a **t-distribution** with $n-1$ degrees of freedom.

▶ The t-distribution is symmetric and bell-shaped.

▶ The t-distribution has heavier tails than the standard normal distribution.

▶ The t-distribution converges to the standard normal distribution as $n \to \infty$.

t-distribution

Exact Confidence Interval for Normal

Notice that T is a pivot.

Let $t_{\alpha/2,n-1}$ be the $\alpha/2$ quantile of the t-distribution with $n-1$ degrees of freedom.

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$$\{-t_{\alpha/2,n-1} < T < t_{\alpha/2,n-1}\} = \left\{ \bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} \right\}$$

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The exact CI for μ is

$$\left(\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right).$$

Exact Confidence Interval for Normal

Proposition

Let $X_1, \ldots, X_n$ be a random sample from a **normal distributed** population with mean μ and variance σ^2. The $100(1 - \alpha)\%$ confidence interval for μ is

$$\left(\bar{X} - t_{\alpha/2, n-1} \frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} \right),$$

where $t_{\alpha/2, n-1}$ is the $\alpha/2$ quantile of the t-distribution with $n - 1$ degrees of freedom and S is the sample standard deviation.

Steps in A Scientific Research

1. ✓ Define the research question. (Your supervisor's job)
2. ✓ Design the experiments. (not in this course)
3. ✓ Collect the data. (Your job)
4. ✓ Probabilistic modeling (Prerequisites)
5. Statistical Inference (this lecture)
   - 5.1 ✓ Point estimation
   - 5.2 ✓ Confidence interval
6. Draw conclusions / Hypothesis Testing (Next lecture)