STAT 423/523 Statistical Methods for Engineers and Scientists

Lecture 1: Review

Chencheng Cai

Washington State University

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- $\mathcal{S} = \{H, T\}$ for flipping a coin
- $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ for tossing a die
- $\mathcal{S} = \mathbb{R}_+$ for measuring the weight of an apple
- $\mathcal{S} = \{\text{recovery}, \text{not recovery}\}$ for a patient taking a drug
- $\mathcal{S} = \{HH, HT, TH, TT\}$ for flipping a coin twice

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- The event that the apple's weight is less than 1: $C = \{x \in \mathbb{R}_+ : x < 1\} = (0, 1)$
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- The **intersection** of two events A and B, denoted by $A \cap B$, is the set of all outcomes that are in both A and B.

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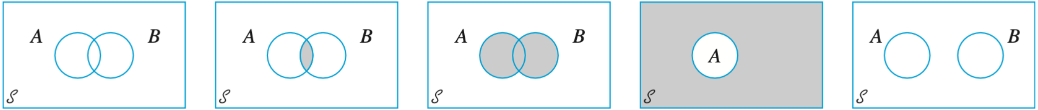
- The event that the sum of two tosses is 7 **or** the first toss is greater than the second is

$$D = A \cup B = \{21, 31, 32, 41, 42, 43, 51, 52, 53, 54, 61, 62, 63, 64, 65, 16, 25, 34\}$$

Venn Diagram

The Venn diagram is a visual representation of events.



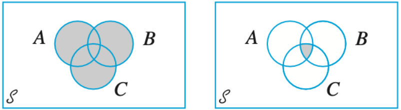
(a) Venn diagram of events A and B

(b) Shaded region is $A \cap B$

(c) Shaded region is $A \cup B$

(d) Shaded region is A'

(e) Mutually exclusive events



(f) Shaded region is $A \cup B \cup C$

(g) Shaded region is $A \cap B \cap C$

For example, it is easy to see that $A \cup B = (A \cap B') \cup (A' \cap B) \cup (A \cap B)$.

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Axioms of probability

1. $P(A) \geq 0$ for any event A.
2. $P(\mathcal{S}) = 1$.
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Remark: A and B are **mutually disjoint** if $A \cap B = \varnothing$.

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Remark: The third axiom includes the finite case by setting $A_i = \varnothing$ for $i > N$.

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Some properties of probability:

- $P(A') = 1 - P(A)$ for any event A. — either A happens or not.
  This can be shown by observing that $A \cap A' = \varnothing$, $A \cup A' = \mathcal{S}$ and by Axiom 3,
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- $P(A \cup B) \leq P(A) + P(B)$ for any events A and B. — the **union bound**.

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- The probability of observing an even prime number is $P(\{2\}) = \frac{1}{6}$.
- Check the inclusion-exclusion principle:

$$P(\{1,3,5\} \cup \{2,3,5\}) + P(\{1,3,5\} \cap \{2,3,5\}) = P(\{1,3,5\}) + P(\{2,3,5\})$$

Conditional Probability

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- $P(A|B)$ is a number between 0 and 1.
- By multiplying $P(B)$ on both sides, we have the **multiplication rule**:

$$P(A \cap B) = P(A|B)P(B).$$

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- This definition of independence can be extended to more than two events: $A_1, A_2, \ldots, A_n$ are **mutually independent** if for any subset $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$

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$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2} = P(A).$$

Therefore, A and B are independent.

Random Variable

For a given sample space $\mathcal{S}$ of some experiment, a **random variable** (rv) is any rule that associates a number with each outcome in $\mathcal{S}$. In mathematical language, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

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- A **discrete random variable** is a random variable that can take on a countable number of values.
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  - The weight of an apple.
  - The time it takes to complete a task.
  - The temperature of a room.

Discrete Random Variable

The **probability distribution** or **probability mass function** (pmf) of a discrete random variable X is defined for every number x by $p(x) = P(X = x)$.

- We use the convention in the textbook that P stands for probability of events and p stands for probability distribution.

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The **cumulative distribution function** (cdf) of a discrete random variable X is defined for every number x by

$$F(x) = P(X \leq x) = \sum_{y: y \leq x} p(y)$$

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The cdf of X is

$$F(5) = P(X \leq 5) = p(2) + p(3) + p(4) + p(5) = \frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{1}{9} = \frac{5}{18}$$

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The **variance** of a discrete random variable X is defined by

$$Var(X) = E[(X - E(X))^2] = E(X^2) - E(X)^2$$

Discrete Random Variable

Let X and Y be two rvs and a and b be two constants. Then

- $E(aX + bY) = aE(X) + bE(Y)$
- $Var(aX) = a^2 Var(X)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab \cdot Cov(X, Y)$

Discrete Random Variable

Common discrete distributions:

- Bernoulli
- Binomial
- Poisson
- Geometric
- Hypergeometric

Continuous Random Variables

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The **cumulative distribution function** (cdf) of a continuous random variable X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

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The **variance** of a continuous random variable X is defined by

$$Var(X) = E[(X - E(X))^2] = E(X^2) - E(X)^2$$

Continuous Random Variables

Let X be a uniform random variable on the interval $[0, 1]$. Then its pdf is $f(x) = 1$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise because for any a and b with $0 \leq a \leq b \leq 1$,

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and the variance of X is

$$Var(X) = E(X^2) - E(X)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Continuous Random Variables

Common continuous distributions:

- Uniform
- Normal
- Exponential
- Gamma
- Beta

Joint Distribution

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The **conditional distribution** of Y given $X = x$ is the pdf of Y given $X = x$:

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}$$

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The **correlation** of X and Y is

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

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- Example: X is a standard normal random variable and Z is a Rademarcher random variable (random ± 1). Let $Y = XZ$. Then X and Y are uncorrelated but not independent.