CISER Causal Inference Workshop

Session 3: Instrumental Variable Methods

Chencheng Cai

Department of Mathematics and Statistics

Washington State University

Mar 31, 2025

Unconfoundedness Assumption

The unconfoundedness assumption between treatment and outcome is:

$$W_i \perp\!\!\!\perp (Y_i(0), Y_i(1)) \mid X_i.$$

- **Problem**: in reality, we may not include all the confounders in X_i.

Unconfoundedness Assumption

The unconfoundedness assumption between treatment and outcome is:

$$W_i \perp\!\!\!\perp (Y_i(0), Y_i(1)) \mid X_i.$$

- **Problem**: in reality, we may not include all the confounders in X_i.
- **Solution**: we design a randomized experiment such that W_i is independent of $(Y_i(0), Y_i(1))$ unconditionally.
- **New problem**: the units may not follow the assigned treatment.

Unconfoundedness Assumption

The unconfoundedness assumption between treatment and outcome is:

$$W_i \perp\!\!\!\perp (Y_i(0), Y_i(1)) \mid X_i.$$

- **Problem**: in reality, we may not include all the confounders in X_i.
- **Solution**: we design a randomized experiment such that W_i is independent of $(Y_i(0), Y_i(1))$ unconditionally.
- **New problem**: the units may not follow the assigned treatment.
- **Solution**: we need to take the compliance issue into account.

The Workflow with Non-compliance

$$Z_i \longrightarrow W_i^{obs} \longrightarrow Y_i^{obs}$$

- Z_i: the treatment assignment (1: treatment, 0: control)
- W_i^{obs}: the observed treatment status (1: treatment, 0: untreated)
- Y_i^{obs}: the observed outcome

The Workflow with Non-compliance

$$Z_i \longrightarrow W_i^{obs} \longrightarrow Y_i^{obs}$$

- Z_i: the treatment assignment (1: treatment, 0: control)
- W_i^{obs}: the observed treatment status (1: treatment, 0: untreated)
- Y_i^{obs}: the observed outcome

Remark:

- Unconfoundedness condition for Z_i and Y_i holds.
- Unconfoundedness condition for W_i and Y_i does not hold.

The Workflow with Non-compliance

$$Z_i \longrightarrow W_i^{obs} \longrightarrow Y_i^{obs}$$

- Z_i: the treatment assignment (1: treatment, 0: control)
- W_i^{obs}: the observed treatment status (1: treatment, 0: untreated)
- Y_i^{obs}: the observed outcome

Remark:

- Unconfoundedness condition for Z_i and Y_i holds.
- Unconfoundedness condition for W_i and Y_i does not hold.

Consequences:

- We cannot estiamte the average causal effects by the difference in means for the **assigned treated** group and the **assigned control** group.
- We cannot estimate the average causal effects by the difference in means for the **observed treated** group and the **observed control** group.

Non-compliance

We view the treatment received W_i a deterministic varaible given the treatment assignment Z_i for unit i.
Then for each unit i, the treatment received W_i has the **potential outcomes**:

$$W_i(Z_i = 1) \quad \text{and} \quad W_i(Z_i = 0)$$

The observed treatment status is:

$$W_i^{obs} = W_i(1)Z_i + W_i(0)(1 - Z_i).$$

Non-compliance

- **Compliers**: $W_i(1) = 1$ and $W_i(0) = 0$.
- **Always-takers**: $W_i(1) = 1$ and $W_i(0) = 1$.
- **Never-takers**: $W_i(1) = 0$ and $W_i(0) = 0$.
- **Defiers**: $W_i(1) = 0$ and $W_i(0) = 1$.

Non-compliance

- **Compliers**: $W_i(1) = 1$ and $W_i(0) = 0$.
- **Always-takers**: $W_i(1) = 1$ and $W_i(0) = 1$.
- **Never-takers**: $W_i(1) = 0$ and $W_i(0) = 0$.
- **Defiers**: $W_i(1) = 0$ and $W_i(0) = 1$.

For convience of notation, we denote:

$$U_i = \begin{cases} c & \text{if } W_i(1) = 1 \text{ and } W_i(0) = 0 \\ a & \text{if } W_i(1) = 1 \text{ and } W_i(0) = 1 \\ n & \text{if } W_i(1) = 0 \text{ and } W_i(0) = 0 \\ d & \text{if } W_i(1) = 0 \text{ and } W_i(0) = 1 \end{cases}$$

Non-compliance

Non-compliance status is **latent**.

- $Z_i = 1$ and $W_i^{obs} = 1$: Compliers or Always-takers.
- $Z_i = 1$ and $W_i^{obs} = 0$: Defiers or Never-takers.
- $Z_i = 0$ and $W_i^{obs} = 1$: Always-takers or Defiers.
- $Z_i = 0$ and $W_i^{obs} = 0$: Compliers or Never-takers.

Potential Outcomes

The potential outcome for unit i under assignment Z_i and received treatment W_i is:

$$Y_i(Z_i, W_i).$$

Potential Outcomes

The potential outcome for unit i under assignment Z_i and received treatment W_i is:

$$Y_i(Z_i, W_i).$$

Because W_i is deterministic given Z_i, the potential outcome only depends on the assigned treatment Z_i that is

$$Y_i(Z_i) = Y_i(Z_i, W_i(Z_i)).$$

Potential Outcomes

Some potential outcomes are not observable under any assignment:

- Compliers: $Y_i(1,0)$ and $Y_i(0,1)$.
- Always-takers: $Y_i(1,0)$, $Y_i(0,0)$.
- Never-takers: $Y_i(1,1)$, $Y_i(0,1)$.
- Defiers: $Y_i(1,1)$, $Y_i(0,0)$.

Potential Outcomes

The observable potential outcomes are:

- Compliers: $Y_i(1) = Y_i(1,1)$ and $Y_i(0) = Y_i(0,0)$.
- Always-takers: $Y_i(1) = Y_i(1,1)$ and $Y_i(0) = Y_i(0,1)$.
- Never-takers: $Y_i(1) = Y_i(1,0)$ and $Y_i(0) = Y_i(0,0)$.
- Defiers: $Y_i(1) = Y_i(1,0)$ and $Y_i(0) = Y_i(0,1)$.

Potential Outcomes

The observable potential outcomes are:

- Compliers: $Y_i(1) = Y_i(1,1)$ and $Y_i(0) = Y_i(0,0)$.
- Always-takers: $Y_i(1) = Y_i(1,1)$ and $Y_i(0) = Y_i(0,1)$.
- Never-takers: $Y_i(1) = Y_i(1,0)$ and $Y_i(0) = Y_i(0,0)$.
- Defiers: $Y_i(1) = Y_i(1,0)$ and $Y_i(0) = Y_i(0,1)$.

Note:

- We can observed two out of the four potential outcomes for each unit under all possible assignments.
- We can observed one out of the two observable potential outcomes for each unit under each assignment.

Intention-to-Treat Effect

By ignoring the non-compliance, we can estimate the effect of the assignment on the outcome by the difference in means:

$$\widehat{\mathsf{ITT}}_Y = \frac{1}{N_t}\sum_{i=1}^{N} Z_i Y_i^{obs} - \frac{1}{N_c}\sum_{i=1}^{N}(1 - Z_i) Y_i^{obs}.$$

Intention-to-Treat Effect

By ignoring the non-compliance, we can estimate the effect of the assignment on the outcome by the difference in means:

$$\widehat{\mathsf{ITT}}_Y = \frac{1}{N_t}\sum_{i=1}^{N} Z_i Y_i^{obs} - \frac{1}{N_c}\sum_{i=1}^{N}(1 - Z_i) Y_i^{obs}.$$

as well as the effect of the assigned treatment on the received treatment:

$$\widehat{\mathsf{ITT}}_W = \frac{1}{N_t}\sum_{i=1}^{N} Z_i W_i^{obs} - \frac{1}{N_c}\sum_{i=1}^{N}(1 - Z_i) W_i^{obs}.$$

N_t and N_c are the number of units assigned with treatment and control respectively.

Intention-to-Treat Effect

Under the **randomization of Z assumption**:

$$Z_i \perp\!\!\!\perp (W_i(0), W_i(1), Y_i(0), Y_i(1)),$$

Both $\widehat{\mathsf{ITT}}_Y$ and $\widehat{\mathsf{ITT}}_W$ are unbiased estimators of the average treatment effect on the assigned treatment.

Intention-to-Treat Effect

Under the **randomization of Z assumption**:

$$Z_i \perp\!\!\!\perp (W_i(0), W_i(1), Y_i(0), Y_i(1)),$$

Both $\widehat{\text{ITT}}_Y$ and $\widehat{\text{ITT}}_W$ are unbiased estimators of the average treatment effect on the assigned treatment.

But, the ITT effect is usually not the causal effect of interest.

Estimands

Use law of total expectation, we can write the true ITT effects in terms of the average effect from different compliance types:

$$
\begin{aligned}
\text{ITT}_Y &= E(Y_i(1) - Y_i(0)) \\
&= E(Y_i(1) - Y_i(0) \mid U_i = c)P(U_i = c) + E(Y_i(1) - Y_i(0) \mid U_i = a)P(U_i = a) \\
&\quad + E(Y_i(1) - Y_i(0) \mid U_i = n)P(U_i = n) + E(Y_i(1) - Y_i(0) \mid U_i = d)P(U_i = d)
\end{aligned}
$$

Estimands

Use law of total expectation, we can write the true ITT effects in terms of the average effect from different compliance types:

$$
\begin{aligned}
\text{ITT}_Y &= E(Y_i(1) - Y_i(0)) \\
&= E(Y_i(1) - Y_i(0) \mid U_i = c)P(U_i = c) + E(Y_i(1) - Y_i(0) \mid U_i = a)P(U_i = a) \\
&\quad + E(Y_i(1) - Y_i(0) \mid U_i = n)P(U_i = n) + E(Y_i(1) - Y_i(0) \mid U_i = d)P(U_i = d)
\end{aligned}
$$

Nowe we check the four terms:

- $Y_i(1) - Y_i(0) \mid U_i = c$ is $Y_i(1,1) - Y_i(0,0) \mid U_i = c$, the causal effect.
- $Y_i(1) - Y_i(0) \mid U_i = a$ is $Y_i(1,1) - Y_i(0,1) \mid U_i = a$, the ITT effect.
- $Y_i(1) - Y_i(0) \mid U_i = n$ is $Y_i(1,0) - Y_i(0,0) \mid U_i = n$, the ITT effect.
- $Y_i(1) - Y_i(0) \mid U_i = d$ is $Y_i(1,0) - Y_i(0,1) \mid U_i = d$, revese causal effect.

Additional Assumptions

Monotonicity Assumption / No-difiers Assumption:

$$D_i(1) \geq D_i(0) \quad \text{for all } i.$$

Additional Assumptions

Monotonicity Assumption / No-difiers Assumption:

$$D_i(1) \geq D_i(0) \quad \text{for all } i.$$

Exclusion Restriction Assumption:

$$Y_i(0) = Y_i(1) \quad \text{for all always-takers and never-takers.}$$

Estimand

Under the randomization of Z assumption, the monotonicity assumption and the exclusion restriction assumption, the ITT effect can be expressed as:

$$\text{ITT}_Y = E(Y_i(1) - Y_i(0) \mid U_i = c)P(U_i = c)$$

Estimand

Under the randomization of Z assumption, the monotonicity assumption and the exclusion restriction assumption, the ITT effect can be expressed as:

$$\text{ITT}_Y = E(Y_i(1) - Y_i(0) \mid U_i = c)P(U_i = c)$$

The part $E(Y_i(1) - Y_i(0) \mid U_i = c)$ is called the **local average treatment effect (LATE)** or the **compliers average treatment effect (CATE)**.

Estimand

The ITT effect on the received treatment can be expressed as:

$$\text{ITT}_W = E(W_i(1) - W_i(0) \mid U_i = c)P(U_i = c) = P(U_i = c)$$

under the previous assumptions.

Estimand

The ITT effect on the received treatment can be expressed as:

$$\text{ITT}_W = E(W_i(1) - W_i(0) \mid U_i = c)P(U_i = c) = P(U_i = c)$$

under the previous assumptions.

Furthermore,

$$\text{CATE} = \frac{ITT_Y}{ITT_W}.$$

Estimator

The instrumental variable (IV) estimator or Wald estimator is:

$$\widehat{\text{CATE}}^{iv} = \frac{\widehat{\text{ITT}}_Y}{\widehat{\text{ITT}}_W}$$

Estimator

The instrumental variable (IV) estimator or Wald estimator is:

$$\widehat{\mathsf{CATE}}^{iv} = \frac{\widehat{\mathsf{ITT}}_Y}{\widehat{\mathsf{ITT}}_W}$$

Intuition:

$$\frac{\Delta Y}{\Delta W} = \frac{\Delta Y/\Delta Z}{\Delta W/\Delta Z}$$

Estimator

The instrumental variable (IV) estimator or Wald estimator is:

$$\widehat{\mathsf{CATE}}^{iv} = \frac{\widehat{\mathsf{ITT}}_Y}{\widehat{\mathsf{ITT}}_W}$$

Intuition:

$$\frac{\Delta Y}{\Delta W} = \frac{\Delta Y / \Delta Z}{\Delta W / \Delta Z}$$

- Randomization assumption: Z affects Y through W.
- Monotonicity assumption: the numerator/denumerator is estimable.
- Exclusion restriction assumption: $\Delta W \neq 0$.

Variance

For the variance of the IV estimator,

- Method 1: delta's method.
- Method 2: bootstrap.
- Method 3: Neyman's formula:

$$\mathrm{Var}\left(\widehat{\mathsf{CATE}}^{iv}\right) = \mathrm{Var}\left(\frac{\widehat{\mathsf{ITT}}_Y - \mathsf{CATE}\cdot\widehat{\mathsf{ITT}}_W}{\widehat{\mathsf{ITT}}_W}\right)$$

The numerator is the difference-in-means estimator for the adjusted outcome:

$$\tilde{Y}_i^{obs} = Y_i^{obs} - \widehat{\mathsf{CATE}}^{iv} W_i^{obs}.$$

Instrumental Variable

An **instrumental variable** is a variable that is correlated with the **received treatment** but not correlated with the **potential outcomes**.

Instrumental Variable

An **instrumental variable** is a variable that is correlated with the **received treatment** but not correlated with the **potential outcomes**. In our previous case:

- Z_i is correlated with W_i^{obs} by removing defiers.
- Z_i is not correlated with $Y_i(1), Y_i(0)$ by randomization assumption.
- Z_i is an instrumental variable for W_i^{obs}.

Regarding the Assumptions

Randomization Assumption:

- Usually holds in randomized experiments.

Exclusion Restriction Assumption:

- Usually holds in a double-blinded experiments.
- May not hold in a single-blinded experiments (placebo-effect).

Monotonicity Assumption:

- Usually holds in a one-sided non-compliance situation.
- Also holds when control group is passively tracked
  – $W_i^{obs} = 0$ for the control group.

Regarding the Assumptions

If we relax the monotonicity assumption, then

$$\text{ITT}_Y = E(Y_i(1) - Y_i(0) \mid U_i = c)P(U_i = c) + E(Y_i(1) - Y_i(0) \mid U_i = d)P(U_i = d)$$
$$\text{ITT}_W = P(U_i = c) - P(U_i = d)$$

Regarding the Assumptions

If we relax the monotonicity assumption, then

$$\mathsf{ITT}_Y = E(Y_i(1) - Y_i(0) \mid U_i = c)P(U_i = c) + E(Y_i(1) - Y_i(0) \mid U_i = d)P(U_i = d)$$
$$\mathsf{ITT}_W = P(U_i = c) - P(U_i = d)$$

The ratio becomes:

$$\frac{\mathsf{ITT}_Y}{\mathsf{ITT}_W} = E(Y_i(1) - Y_i(0) \mid U_i = c)\frac{P(U_i = c)}{P(U_i = c) - P(U_i = d)} + E(Y_i(1) - Y_i(0) \mid U_i = d)\frac{P(U_i = d)}{P(U_i = c) - P(U_i = d)}$$

Regarding the Assumptions

If we relax the monotonicity assumption, then

$$\text{ITT}_Y = E(Y_i(1) - Y_i(0) \mid U_i = c)P(U_i = c) + E(Y_i(1) - Y_i(0) \mid U_i = d)P(U_i = d)$$
$$\text{ITT}_W = P(U_i = c) - P(U_i = d)$$

The ratio becomes:

$$\frac{\text{ITT}_Y}{\text{ITT}_W} = E(Y_i(1) - Y_i(0) \mid U_i = c)\frac{P(U_i = c)}{P(U_i = c) - P(U_i = d)} + E(Y_i(1) - Y_i(0) \mid U_i = d)\frac{P(U_i = d)}{P(U_i = c) - P(U_i = d)}$$

It is the weighted average of the average causal effect on the compliers and the average causal effect on the defiers.

Weak IV

$$\text{CATE} = \frac{\text{ITT}_Y}{\text{ITT}_W}$$

The finite performance of the IV estimator could be poor if

$$\text{ITT}_W \approx 0.$$

- Higher probability for $\widehat{\text{ITT}}_W$ to be close to 0 when the sample size is small.
- The distribution of $\widehat{\text{CATE}}^{iv}$ is not normal.

Weak IV

$$\text{CATE} = \frac{\text{ITT}_Y}{\text{ITT}_W}$$

The finite performance of the IV estimator could be poor if

$$\text{ITT}_W \approx 0.$$

- Higher probability for $\widehat{\text{ITT}}_W$ to be close to 0 when the sample size is small.
- The distribution of $\widehat{\text{CATE}}^{iv}$ is not normal.

Such instrumental variables that are weakly correlated with the treatment are called **weak instrumental variables**.

Weak IV

For hypothesis testing,

$$H_0 : \mathsf{CATE} = 0 \quad \Longleftrightarrow \quad H_0' : \mathsf{ITT}_Y = 0.$$

testing H_0' is simpler.

Weak IV

For hypothesis testing,

$$H_0 : \text{CATE} = 0 \quad \Longleftrightarrow \quad H_0' : \text{ITT}_Y = 0.$$

testing H_0' is simpler.

For confidence interval,

- Find values for b such that we cannot reject $H_0 : \text{CATE} = b$ using Wald test.
- Similar idea to the Anderson-Rubin confidence interval and the profiled likelihood ratio confidence interval.

Connection to Econometric Methods

Consider the following **structural equation** to model the treatment effect:

$$Y_i^{obs} = \alpha + \tau \cdot W_i^{obs} + \epsilon_i.$$

Connection to Econometric Methods

Consider the following **structural equation** to model the treatment effect:

$$Y_i^{obs} = \alpha + \tau \cdot W_i^{obs} + \epsilon_i.$$

- Constant causal effect τ.
- Exclusion restriction assumption for all the units.

Connection to Econometric Methods

Consider the following **structural equation** to model the treatment effect:

$$Y_i^{obs} = \alpha + \tau \cdot W_i^{obs} + \epsilon_i.$$

- Constant causal effect τ.
- Exclusion restriction assumption for all the units.
- ϵ_i is **defined** to be
$$Y_i^{obs} - \alpha - \tau \cdot W_i^{obs}.$$
- W_i^{obs} is correlated with ϵ_i. That is W_i^{obs} is **endogenous**.
- **NOT** a regression problem.

Connection to Econometric Methods

$$Y_i^{obs} = \alpha + \tau \cdot W_i^{obs} + \epsilon_i.$$

Z_i is an **instrumental variable** for W_i^{obs} because:

- Z_i is correlated with W_i^{obs}.
- Z_i is not correlated with ϵ_i.

Connection to Econometric Methods

$$Y_i^{obs} = \alpha + \tau \cdot W_i^{obs} + \epsilon_i.$$

Z_i is an **instrumental variable** for W_i^{obs} because:

- Z_i is correlated with W_i^{obs}.
- Z_i is not correlated with ϵ_i.

A typical utilization of the instrumental variable is to estimate the causal effect τ by the **two-stage least squares (TSLS)** method.

Connection to Econometric Methods

Because Z_i is uncorrelated with ϵ_i,

$$E(Y_i^{obs} \mid Z_i) = \alpha + \tau E(W_i^{obs} \mid Z_i)$$

Connection to Econometric Methods

Because Z_i is uncorrelated with ϵ_i,

$$E(Y_i^{obs} \mid Z_i) = \alpha + \tau E(W_i^{obs} \mid Z_i)$$

Then

$$Y_i^{obs} = \alpha + \tau \cdot E(W_i^{obs} \mid Z_i) + \underbrace{\tau W_i^{obs} - \tau \cdot E(W_i^{obs} \mid Z_i) + \epsilon_i}_{\eta_i}$$

Connection to Econometric Methods

Because Z_i is uncorrelated with ϵ_i,

$$E(Y_i^{obs} \mid Z_i) = \alpha + \tau E(W_i^{obs} \mid Z_i)$$

Then

$$Y_i^{obs} = \alpha + \tau \cdot E(W_i^{obs} \mid Z_i) + \underbrace{\tau W_i^{obs} - \tau \cdot E(W_i^{obs} \mid Z_i) + \epsilon_i}_{\eta_i}$$

By the randomization assumption, $E(W_i^{obs} \mid Z_i)$ and η_i are uncorrelated.

Connection to Econometric Methods

Because Z_i is uncorrelated with ϵ_i,

$$E(Y_i^{obs} \mid Z_i) = \alpha + \tau E(W_i^{obs} \mid Z_i)$$

Then

$$Y_i^{obs} = \alpha + \tau \cdot E(W_i^{obs} \mid Z_i) + \underbrace{\tau W_i^{obs} - \tau \cdot E(W_i^{obs} \mid Z_i) + \epsilon_i}_{\eta_i}$$

By the randomization assumption, $E(W_i^{obs} \mid Z_i)$ and η_i are uncorrelated.

τ can be estiamted by the least squares method after we know the value of $E(W_i^{obs} \mid Z_i)$.

Connection to Econometric Methods

The value of $E(W_i^{obs} \mid Z_i)$ can be estimated by (the first stage):

$$E(W_i^{obs} \mid Z_i) = \pi_0 + \pi_1 Z_i,$$

In our case, $\pi = 0 = 0$ and π_1 is the proportion of compliers.

Connection to Econometric Methods

The value of $E(W_i^{obs} \mid Z_i)$ can be estimated by (the first stage):

$$E(W_i^{obs} \mid Z_i) = \pi_0 + \pi_1 Z_i,$$

In our case, $\pi = 0 = 0$ and π_1 is the proportion of compliers.

In the second stage, we fit

$$Y_i^{obs} = \alpha + \tau \cdot E(W_i^{obs} \mid Z_i) + \eta_i = \alpha + \tau \hat{\pi}_1 Z_i + \eta_i.$$

Connection to Econometric Methods

The value of $E(W_i^{obs} \mid Z_i)$ can be estimated by (the first stage):

$$E(W_i^{obs} \mid Z_i) = \pi_0 + \pi_1 Z_i,$$

In our case, $\pi = 0 = 0$ and π_1 is the proportion of compliers.

In the second stage, we fit

$$Y_i^{obs} = \alpha + \tau \cdot E(W_i^{obs} \mid Z_i) + \eta_i = \alpha + \tau \hat{\pi}_1 Z_i + \eta_i.$$

$\hat{\tau}$ is the ratio of the two stages' regression coefficients.

IV + Bayesian Methods

- A multinomial logistic regression model for the compliance categories.
- Gaussian model for the potential outcomes.
- With a prior on all the parameters, samples for the missing potential outcomes can be drawn from the posterior distribution:

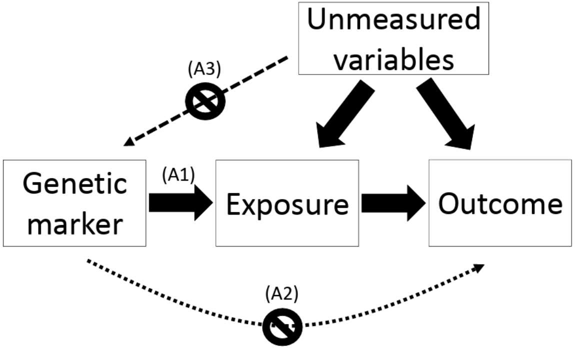
$$f(\boldsymbol{Y}^{mis}, \boldsymbol{W}^{mis} \mid \boldsymbol{Y}^{obs}, \boldsymbol{W}^{obs}, \boldsymbol{X}, \boldsymbol{Z})$$

- Causal effects can be estimated by the imputed values.

Imbens & Rubin (1997). "Bayesian Inference for Causal Effects in Randomized Experiments with Noncompliance." AoS.

Mendelian Randomization

Mendelian randomization with many (possibly invalid) instrumental variables.



Kang, Zhang, Cai,& Small (2016). "Instrumental Variables Estimation With Some Invalid Instruments and its Application to Mendelian Randomization." JASA

ML + IV

Use ML to replace the regression model in TSLS.

Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, Newey, & Robins (2018). "Double/debiased machine learning for treatment and structural parameters." The Econometrics Journal

Thank you for joining the workshop!

Contact:

Me	chencheng.cai@wsu.edu
CISER	ciser.info@wsu.edu